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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

... ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

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On Korn's Inequality

L. E. PAYNE & H. F. WEINBERGER

Communicated by R. A. TOUPIN

1. Introduction

Let $\vec{u}(x_1, x_2, x_3)$ be a continuously differentiable vector field on a region R with boundary B. If an isotropic homogeneous elastic medium in R is subjected to an infinitesimal displacement proportional to $\vec{u}$, the classical strain energy is proportional to

$$E(\vec{u}) = S(\vec{u}) + \frac{\sigma}{1-2\sigma} \iiint\limits_R |\operatorname{div} \vec{u}|^2 \, dx \tag{1.1}$$

where

$$S(\vec{u}) = \tfrac{1}{4} \iiint\limits_R (u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) \, dx. \tag{1.2}$$

Here u_1, u_2, u_3 are the components of $\vec{u}$ in the rectangular coordinates (x_1, x_2, x_3), dx is the element of volume, the constant σ is Poisson's ratio, and the symbol ${}_{i,j}$ indicates differentiation with respect to x_j. We use the summation convention throughout.

We introduce the Dirichlet norm

$$D(\vec{u}) = \iiint\limits_R u_{i,j} u_{i,j} \, dx. \tag{1.3}$$

KORN'S inequality asserts that if $\vec{u}$ satisfies the integral conditions

$$\iiint\limits_R (u_{i,j} - u_{j,i}) \, dx = 0, \qquad i, j = 1, 2, 3, \tag{1.4}$$

if

$$\sigma > -1, \tag{1.5}$$

and if R is sufficiently regular, there exists a constant K_0 depending only upon σ and R such that

$$D(\vec{u}) \leqq K_0 E(\vec{u}). \tag{1.6}$$

It is easily seen that

$$0 \leqq \iiint\limits_R |\operatorname{div} \vec{u}|^2 \, dx \leqq 3 S(\vec{u}). \tag{1.7}$$

Thus if (1.6) holds for $\sigma = 0$, *i.e.*, if

$$D(\vec{u}) \leqq K_1 S(\vec{u}), \tag{1.8}$$

then (1.6) clearly holds for all $\sigma > -1$, with $K_0 = K_1$ for $\sigma \geqq 0$ and $K_0 = (1+\sigma) K_1/(1-2\sigma)$ for $-1 < \sigma \leqq 0$. (For negative values of σ we have made use of (1.7).)

The inequality (1.6) is useful in proving the existence and uniqueness of the solution of the second boundary-value problem of classical elasticity. An explicit knowledge of the constant K_0 also leads to explicit arbitrarily close bounds for the solution. This follows from results of [7] if B is star-shaped with respect some point in R, and from [8] if R lies exterior to B.

We now let

$$R(\vec{u}) = \tfrac{1}{4} \iiint_R (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i})\, dx. \tag{1.9}$$

It is easily shown that the inequalities (1.8),

$$R(\vec{u}) \leqq (1 - K_1^{-1})\, D(\vec{u}) \tag{1.10}$$

and

$$R(\vec{u}) \leqq (K_1 - 1)\, S(\vec{u}) \tag{1.11}$$

are completely equivalent. KORN [4, 5], FRIEDRICHS [3], and BERNSTEIN & TOUPIN [1] define the constant $(K_1 - 1)$ of (1.11) to be the Korn constant K. It is clear that $K_1 \geqq 1$.

KORN's inequality was first proved by A. KORN [4, 5] by means of integral equations on the boundary. A more direct proof involving integration over the interior of R was given by K. O. FRIEDRICHS [3]. In a recent paper in this journal, B. BERNSTEIN & R. TOUPIN [1] gave a simplified version of the FRIEDRICHS proof which permitted them to prove the inequality (1.8) with the explicit value

$$K_1 = 21 \tag{1.12}$$

for the case when R is a sphere*.

In this paper we reverse the procedure of BERNSTEIN & TOUPIN. We first prove the inequality (1.8) for the sphere by explicitly determining the spectrum of the quadratic form $D(\vec{u})$ with respect to $S(\vec{u})$, thus showing that it is bounded. In this way we find the best possible constant

$$K_1 = \tfrac{56}{13} \tag{1.13}$$

in (1.8) for the case of the sphere. We also find the best value of K_0 for any $\sigma > -1$ in the inequality (1.6) for the sphere.

In Section 3 we extend KORN's inequality to any region R that can be mapped onto the sphere by a twice differentiable mapping.

In Section 4 we show that if KORN's inequality holds on several domains, it also holds in their union. In this way we obtain KORN's inequality on a rather general class of regions R.

In Section 5 we obtain for the two-dimensional circular region the best possible constant

$$K_1 = 4.$$

2. The sphere

Let R be the unit sphere centered at the origin. Let $\vec{u}$ be a continuously differentiable vector field satisfying the normalization conditions (1.3). It is

* It is easily seen that (1.8) is invariant under a uniform dilatation. Thus, the constant K_1 for a sphere is independent of its radius.

easily seen that under these conditions both $S(\vec{u})$ and $D(\vec{u})$ are positive definite, providing we identify vector fields differing only by a constant. We seek the spectrum of $D(\vec{u})$ with respect to $S(\vec{u})$. The inequality (1.8) asserts that this spectrum is bounded. It follows immediately from integration by parts that if

$$\operatorname{div}\vec{v}=0 \quad \text{in } R \tag{2.1}$$

and if

$$\vec{v}=0 \quad \text{on } B, \tag{2.2}$$

then

$$D(\vec{v})=2S(\vec{v}). \tag{2.3}$$

Thus, 2 is an eigenvalue of infinite multiplicity.

Also, if ψ is any twice differentiable function, then

$$D(\operatorname{grad}\psi)=S(\operatorname{grad}\psi). \tag{2.4}$$

Thus, 1 is also an eigenvalue of infinite multiplicity.

To obtain the remainder of the spectrum, we derive the Euler equation in the usual manner. We obtain the differential equation

$$(2-K)\,u_{j,ii}-K\,u_{i,ij}=0 \quad \text{in } R; \qquad j=1,2,3 \tag{2.5}$$

and the boundary conditions

$$[(2-K)\,u_{j,i}-K\,u_{i,j}]\,x_i=0 \quad \text{on } B; \qquad j=1,2,3. \tag{2.6}$$

Here K is the eigenvalue parameter. We have used the fact that the radius vector $\vec{r}$ with components x_i is a normal vector on B.

Applying the operator $\vec{r}\cdot\operatorname{curl}$ to (2.5), we find for $K\neq 2$

$$\Delta(\vec{r}\cdot\operatorname{curl}\vec{u})=0. \tag{2.7}$$

Since the operator $\vec{r}\cdot\operatorname{curl}$ involves only tangential differentiation on the sphere, we can apply it to (2.6) to obtain

$$(2-K)\,r\frac{\partial}{\partial r}(\vec{r}\cdot\operatorname{curl}\vec{u})+K\,\vec{r}\cdot\operatorname{curl}\vec{u}=0 \quad \text{on } B. \tag{2.8}$$

It follows from (2.7) that the left-hand side of (2.8) is harmonic in R. Hence it vanishes identically. Thus, $\vec{r}\cdot\operatorname{curl}\vec{u}$ is a regular harmonic function which is homogeneous of degree $-K(2-K)^{-1}$ in R. Either it vanishes identically, or this degree must be a positive integer. (Since $\vec{r}\cdot\operatorname{curl}\vec{u}=\frac{1}{2}\operatorname{div}[(r^2-1)\operatorname{curl}\vec{u}]$, its integral vanishes, so that $\vec{r}\cdot\operatorname{curl}\vec{u}$ cannot be a constant other than zero.) We thus find the eigenvalues

$$K=\frac{2n}{n-1}, \qquad n=2,3,\ldots \tag{2.9}$$

with the corresponding eigenvectors

$$\vec{u}=\vec{r}\times\operatorname{grad}P^{(n)}. \tag{2.10}$$

Here $P^{(n)}$ is any spherical harmonic of degree n. That is, it is a homogeneous polynomial in the x_i of degree n and is harmonic. The function (2.10) is easily seen to satisfy $\vec{r}\cdot\operatorname{curl}\vec{u}=-n(n+1)\,P^{(n)}$ as well as (2.5) and (2.6).

Note that the case $n=1$ in (2.9) is eliminated by condition (1.4).

If we have any function of the form

$$\vec{u} = \vec{r} \times \operatorname{grad} \eta \tag{2.11}$$

where

$$\Delta \eta = 0, \tag{2.12}$$

we may expand η in spherical harmonics:

$$\eta \sim \sum_{n=1}^{\infty} a_n P^{(n)}. \tag{2.13}$$

Normalizing the $P^{(n)}$ so that

$$D(\vec{r} \times \operatorname{grad} P^{(n)}) = 1, \tag{2.14}$$

we find that if $D(\vec{r} \times \operatorname{grad} \eta)$ exists,

$$\begin{aligned} D(\vec{r} \times \operatorname{grad} \eta) &= \sum_{n=1}^{\infty} |a_n|^2, \\ S(\vec{r} \times \operatorname{grad} \eta) &= \sum_{n=2}^{\infty} \frac{1}{2} \left(\frac{n-1}{n}\right) |a_n|^2. \end{aligned} \tag{2.15}$$

If $\vec{u}$ is subjected to the condition (1.4), we have

$$\iiint_R \operatorname{curl} [\vec{r} \times \operatorname{grad} \eta] \, d x^1 \, d x^2 \, d x^3 = - \iint_B \left(\eta + r \frac{\partial \eta}{\partial r}\right) \vec{r} \, d S = 0, \tag{2.16}$$

which implies that $a_1 = 0$. Consequently, we find that

$$D(\vec{r} \times \operatorname{grad} \eta) \leqq 4 S(\vec{r} \times \operatorname{grad} \eta). \tag{2.17}$$

Finally we consider the eigenfunctions of (2.5), (2.6) for which $\vec{r} \cdot \operatorname{curl} \vec{u} = 0$. This means that $\vec{u}$ can be expressed as the sum of a gradient and a multiple of $\vec{r}$. Taking the divergence and curl of (2.5) shows that if $K \neq 1, 2$ both $\operatorname{div} \vec{u}$ and $\operatorname{curl} \vec{u}$ are harmonic. It follows that $\vec{u}$ can be expressed in terms of three harmonic functions φ, ϱ and τ by

$$\vec{u} = \operatorname{grad} [\varphi + r^2 \varrho] + (r^2 - 1) \operatorname{grad} \tau, \tag{2.18}$$

$$\Delta \varphi = \Delta \varrho = \Delta \tau = 0. \tag{2.19}$$

Inserting this result into (2.5), we find that (aside from an unessential additive constant)

$$2(1-K)\left(2r \frac{\partial}{\partial r} + 3\right) \varrho + \left[(4 - 3K) r \frac{\partial}{\partial r} + 2 - K\right] \tau = 0. \tag{2.20}$$

The normal component of (2.6) gives, if $K \neq 1$,

$$r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} - 1\right) \varphi + \left(r \frac{\partial}{\partial r} + 2\right)\left(r \frac{\partial}{\partial r} + 1\right) \varrho + 2r \frac{\partial \tau}{\partial r} = 0 \tag{2.21}$$

on B. Since the left-hand side is harmonic, (2.21) holds in R. Similarly we find from the tangential component of (2.6) that

$$(1-K)\left(r \frac{\partial}{\partial r} - 1\right) \varphi + (1-K)\left(r \frac{\partial}{\partial r} + 1\right) \varrho + (2-K) \tau = 0 \tag{2.22}$$

in R.

Since the equations (2.20), (2.21), (2.22) involve only the differential operator $r\frac{\partial}{\partial r}$, the harmonic functions φ, ϱ and τ are multiples of a spherical harmonic $P^{(n)}$. We find the eigenvalues

$$K = \frac{2(n+1)(2n+1)}{n^2+n+1}, \qquad n = 1, 2, \ldots \tag{2.23}$$

with the corresponding eigenvectors

$$\vec{u} = \operatorname{grad}\left[n\{2n+3-r^2(2n+1)\}P^{(n)}\right] + (r^2-1)\operatorname{grad}\left[(3n^2+5n+1)P^{(n)}\right]. \tag{2.24}$$

Any linear combination $\vec{w}$ of these eigenvectors can be written in terms of a harmonic function ϱ as

$$\begin{aligned}\vec{w} = \operatorname{grad}&\left[\left\{2+(1-r^2)\left(2r\frac{\partial}{\partial r}+1\right)\right\}r\frac{\partial\varrho}{\partial r}\right] + \\ &+ (r^2-1)\operatorname{grad}\left[\left(3\left(r\frac{\partial}{\partial r}\right)^2 + 5r\frac{\partial}{\partial r}+1\right)\varrho\right].\end{aligned} \tag{2.25}$$

Conversely, if $\vec{w}$ is of the form (2.25) and $D(\vec{u})$ exists, we may expand ϱ in spherical harmonics

$$\varrho \sim \sum_{n=1}^{\infty} a_n P^{(n)}. \tag{2.26}$$

Then

$$\begin{aligned} S(\vec{w}) &= \sum_{n=1}^{\infty} a_n^2 S(\vec{u}^{(n)}), \\ D(\vec{w}) &= \sum_{n=1}^{\infty} a_n^2 D(\vec{u}^{(n)}) = \sum_{n=1}^{\infty} a_n^2 \frac{2(n+1)(2n+1)}{n^2+n+1} S(\vec{u}^{(n)}), \end{aligned} \tag{2.27}$$

where $\vec{u}^{(n)}$ denotes the eigenvector (2.24). Both series in (2.27) converge. Noting that the largest of the numbers (2.23) is given when $n=3$, we find that for $\vec{w}$ of the form (2.25)

$$D(\vec{w}) \leqq \tfrac{56}{13} S(\vec{w}). \tag{2.28}$$

Now if $\vec{u}$ can be expressed in the form

$$\vec{u} = \vec{v} + \operatorname{grad}\psi + \vec{r}\times\operatorname{grad}\eta + \vec{w} \tag{2.29}$$

where

$$\begin{aligned} \operatorname{div}\vec{v} &= 0 \quad \text{in } R, \\ \vec{v} &= 0 \quad \text{on } B, \\ \Delta\eta &= 0 \quad \text{in } R, \end{aligned} \tag{2.30}$$

and w is given by (2.25) with

$$\Delta\varrho = 0 \quad \text{in } R, \tag{2.31}$$

it is easily seen by integration by parts that

$$\begin{aligned} S(\vec{u}) &= S(\vec{v}) + S(\operatorname{grad}\psi) + S(\vec{r}\times\operatorname{grad}\eta) + S(\vec{w}), \\ D(\vec{u}) &= D(\vec{v}) + D(\operatorname{grad}\psi) + D(\vec{r}\times\operatorname{grad}\eta) + D(\vec{w}). \end{aligned} \tag{2.32}$$

That is, the four components are orthogonal in the sense of the norms $[S(\vec{u})]^{\frac{1}{2}}$ and $[D(\vec{u})]^{\frac{1}{2}}$.

It is easily verified that $\vec{v}$, $\operatorname{grad}\psi$, and $\vec{w}$ satisfy the condition (1.4) automatically. Hence (1.4) implies (2.16).

Thus, it follows from (2.3), (2.4), (2.17), and (2.28) that

$$D(\vec{u}) \leqq \tfrac{56}{13}\, S(\vec{u}) \tag{2.33}$$

for all $\vec{u}$ of the form (2.29). We shall show that any sufficiently differentiable vector field $\vec{u}$ is expressible in the form (2.29).

Applying the operator $\vec{r}\cdot\operatorname{curl}$ to (2.29) on the boundary B, we find the boundary condition

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}+1\right)\eta = \vec{r}\cdot\operatorname{curl}\vec{u}. \tag{2.34}$$

Since the left-hand side is harmonic, it (and hence η) is determined by this boundary condition.

Since $\vec{r}\cdot\operatorname{curl}[\vec{u}-\vec{r}\times\operatorname{grad}\eta]$ vanishes on B, the tangential components of $\vec{u}-\vec{r}\times\operatorname{grad}\eta$ coincide with the tangential derivatives of a function f on B. That is, there exist functions f and g such that

$$\vec{u}-\vec{r}\times\operatorname{grad}\eta = \operatorname{grad} f + g\,\vec{r} \quad \text{on } B. \tag{2.35}$$

Using the form (2.25) of $\vec{w}$ and the fact that $\vec{v}$ vanishes on B, we find that

$$\begin{gathered}\psi+\left[2+(1-r^2)\left(2r\frac{\partial}{\partial r}+1\right)\right]r\frac{\partial\varrho}{\partial r}=f\\ \frac{\partial}{\partial r}\left\{\psi+\left[2+(1-r^2)\left(2r\frac{\partial}{\partial r}+1\right)\right]r\frac{\partial\varrho}{\partial r}\right\}=\frac{\partial f}{\partial r}+g \quad \text{on } B.\end{gathered} \tag{2.36}$$

Taking the Laplacian of the divergence of (2.29) we find that

$$\Delta\Delta\left\{\psi+\left[2+(1-r^2)\left(2r\frac{\partial}{\partial r}+1\right)\right]r\frac{\partial\varrho}{\partial r}\right\}=\Delta\operatorname{div}\vec{u}. \tag{2.37}$$

The differential equation (2.37) together with the boundary conditions (2.36) serves to determine the function in braces. Taking the divergence of (2.29), we find

$$2\left(3\left(r\frac{\partial}{\partial r}\right)^2+5r\frac{\partial}{\partial r}+1\right)r\frac{\partial\varrho}{\partial r}=\operatorname{div}\vec{u}-\Delta\left\{\psi+\left[2+(1-r^2)\left(2r\frac{\partial}{\partial r}+1\right)\right]r\frac{\partial\varrho}{\partial r}\right\}. \tag{2.38}$$

Since the right-hand side is already determined and is harmonic by (2.37), this equation determines the harmonic function ϱ. This then determines ψ. The vector field $\vec{v}$ is now obtained by transposing in (2.29). It automatically satisfies (2.30).

Thus we have shown that (2.33) holds for sufficiently differentiable vector fields $\vec{u}$. Hence it holds for limits of such fields in the sense of the norm $[D(\vec{u})]^{\frac{1}{2}}$. These are precisely the vector fields with Dirichlet integrable components.

The inequality (2.33) is sharp in the sense that equality holds for $\vec{u}$ of the form (2.24) with $n=3$.

Remark. In the same manner we can obtain a sharp inequality

$$D(\vec{u}) \leqq K_0\, E(\vec{u}) \tag{2.39}$$

with $E(\vec{u})$ defined by (1.1) for any $\sigma > -1$. We find that

$$K_0 = \begin{cases} 4, & \sigma \geqq \frac{1}{14} \\ \frac{1}{26+14\sigma}\left[69-42\sigma+\{1849-1540\sigma+4900\sigma^2\}^{\frac{1}{2}}\right], & -\frac{1}{25} \leqq \sigma \leqq \frac{1}{14} \\ \frac{1}{14+10\sigma}\left[37-20\sigma+\{529-400\sigma+1600\sigma^2\}^{\frac{1}{2}}\right], & -\frac{1}{4} \leqq \sigma \leqq -\frac{1}{25} \\ \frac{1}{2+2\sigma}\left[5-2\sigma+\{9-4\sigma+36\sigma^2\}^{\frac{1}{2}}\right], & -1 < \sigma \leqq -\frac{1}{4}. \end{cases} \tag{2.40}$$

The different functions of σ correspond to different types of eigenvectors for which the maximum eigenvalue is attained. In particular for $\sigma \geqq \frac{1}{14}$ equality is attained with $\vec{u} = \vec{r} \times \operatorname{grad} P^{(2)}$.

3. Smooth simply connected regions

Let

$$y^i = y^i(x_1, x_2, x_2) \tag{3.1}$$

map the given region R onto the unit sphere $y^i y^i \leqq 1$ when the x_i and y_i are considered as Cartesian coordinates. Let the functions y^i have bounded first and second derivatives, and let the Jacobian determinant be bounded away from zero. Then we can consider the y^i as curvilinear coordinates on R.

If the vector field $\vec{u}$ has the components u_i in the y-coordinates,

$$\begin{aligned} S(\vec{u}) &= \tfrac{1}{4} \iiint_R \sqrt{g}\, g^{ik} g^{jl} (u_{i|j} + u_{j|i})(u_{k|l} + u_{l|k})\, dy^1 dy^2 dy^3 \\ D(\vec{u}) &= \iiint_R \sqrt{g}\, g^{ik} g^{jl} u_{i|j} u_{k|l}\, dy^1 dy^2 dy^3 \end{aligned} \tag{3.2}$$

where

$$u_{i|j} = \frac{\partial u_i}{\partial y^j} - \Gamma_{ij}^p u_p \tag{3.3}$$

denotes the covariant derivative, and the Γ_{ij}^p are the Christoffel symbols [7]. The conjugate metric tensor g^{ij} is defined by

$$g^{ij} = \frac{\partial y^i}{\partial x^p} \frac{\partial y^j}{\partial x^p}, \tag{3.4}$$

and g is the reciprocal of its determinant.

We let

$$\begin{aligned} S_0(\vec{u}) &= \frac{1}{4} \iiint \left(\frac{\partial u_i}{\partial y^j} + \frac{\partial u_j}{\partial y^i}\right)\left(\frac{\partial u_i}{\partial y^j} + \frac{\partial u_j}{\partial y^i}\right) dy^1 dy^2 dy^3, \\ D_0(\vec{u}) &= \iiint \frac{\partial u_i}{\partial y^j} \frac{\partial u_i}{\partial y^j}\, dy^1 dy^2 dy^3, \end{aligned} \tag{3.5}$$

the integrals being over the region $y^i y^i \leqq 1$. We have shown that if u_i satisfies the conditions

$$\iiint_{y^i y^i \leqq 1} \left(\frac{\partial u_i}{\partial y^j} - \frac{\partial u_j}{\partial y^i}\right) dy^1 dy^2 dy^3 = 0, \qquad i, j = 1, 2, 3, \tag{3.6}$$

then

$$D_0(\vec{u}) \leqq \tfrac{56}{13} S_0(\vec{u}). \tag{3.7}$$

Since the mapping (3.1) is assumed to be non-singular and to have bounded second derivatives, there are positive constants α, β, and γ such that

$$\alpha\,\xi_i\,\xi_i \leqq g^{\frac{1}{2}}\,g^{ij}\,\xi_i\,\xi_j \leqq \beta\,\xi_i\,\xi_i \tag{3.8}$$

and

$$\sqrt{g}\,g^{ik}\,g^{jl}\,\Gamma^p_{ij}\,\Gamma^q_{kl}\,\xi_p\,\xi_q \leqq \gamma^2\,\xi_i\,\xi_i \tag{3.9}$$

for all vectors ξ_i and all points of R. It then follows with the aid of the triangle inequality that

$$\begin{aligned} [S(\vec{u})]^{\frac{1}{2}} &\geqq \alpha\,[S_0(\vec{u})]^{\frac{1}{2}} - \gamma\Big[\iiint\limits_{y^i y^i \leqq 1} u_i\,u_i\,d\,y^1\,d\,y^2\,d\,y^3\Big]^{\frac{1}{2}},\\ [D(\vec{u})]^{\frac{1}{2}} &\leqq \beta\,D_0(\vec{u})^{\frac{1}{2}} + \gamma\Big[\iiint\limits_{y^i y^i \leqq 1} u_i\,u_i\,d\,y^1\,d\,y^2\,d\,y^3\Big]^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

It is well known [2] that the quadratic form

$$\iiint\limits_{y^i y^i \leqq 1} \varphi^2\,d\,y^1\,d\,y^2\,d\,y^3 \tag{3.11}$$

is completely continuous with respect to the Dirichlet integral

$$\iiint\limits_{y^i y^i \leqq 1} \frac{\partial\varphi}{\partial y^i}\,\frac{\partial\varphi}{\partial y^i}\,d\,y^1\,d\,y^2\,d\,y^3. \tag{3.12}$$

This means that a finite number of linear functionals $L_\nu(\varphi)$, $\nu = 1, \ldots, N$ can be found such that the integral of φ^2 is less than any positive constant ε^2 times the integral (3.12) whenever $L_\nu(\varphi) = 0$, $\nu = 1, \ldots, N$. If we choose

$$\varepsilon < \alpha\left[\tfrac{13}{56}\right]^{\frac{1}{2}} \tag{3.13}$$

and impose these constraints on all the components u_i, and if in addition the u_i satisfy (3.6), we have from (3.7) and (3.10)

$$D(\vec{u}) \leqq \frac{56\,(\beta + \varepsilon\,\gamma)^2}{(\sqrt{13}\alpha - \sqrt{56}\,\varepsilon)^2}\,S(\vec{u}). \tag{3.14}$$

Thus $D(\vec{u})$ is bounded with respect to $S(\vec{u})$ on the orthogonal complement of a finite-dimensional subspace A of the Hilbert space of vector fields satisfying (1.4) with norm $[S(\vec{u})]^{\frac{1}{2}}$. But $D(\vec{u})$ is also bounded on the finite-dimensional space A. Otherwise there would be a vector field $\vec{u}$ with Dirichlet integrable components satisfying (1.4) for which $S(\vec{u}) = 0$, $D(\vec{u}) = 1$. But $S(\vec{u}) = 0$ implies $u =$ constant and hence $D(\vec{u}) = 0$. Thus, there is a constant K_2 such that

$$D(\vec{w}) \leqq K_2\,S(\vec{w}) \tag{3.15}$$

for $\vec{w}$ in A. Denoting the constant appearing on the right of (3.14) by K_1, we decompose any $\vec{u}$ into a vector $\vec{v}$ in the orthogonal complement of A and $\vec{w}$ in A. Then we find, using SCHWARZ's inequality,

$$\begin{aligned} D(\vec{u}) = D(\vec{v} + \vec{w}) &\leqq \Big(1 + \frac{K_2}{K_1}\Big)D(\vec{v}) + \Big(1 + \frac{K_1}{K_2}\Big)D(\vec{w})\\ &\leqq (K_1 + K_2)\,[S(\vec{v}) + S(\vec{w})]\\ &= (K_1 + K_2)\,S(\vec{u}). \end{aligned} \tag{3.16}$$

Thus, a Korn inequality holds for any domain R that can be mapped onto a sphere by a non-singular mapping with bounded second derivatives.

4. More general regions

Let R be the union of regions $R_1, R_2, \ldots, R_n$ on which KORN'S inequality holds. If $\vec{u}$ is subjected to the conditions (1.4) for each R_ν, we have, in the obvious notation

$$D_\nu(\vec{u}) \leqq K_\nu S_\nu(\vec{u}).$$

Hence

$$D(\vec{u}) \leqq \sum D_\nu(\vec{u}) \leqq \sum K_\nu S_\nu(u) \leqq \sum K_\nu S(\vec{u}). \tag{4.1}$$

Thus, $D(\vec{u})$ is bounded with respect to $S(\vec{u})$ when $\vec{u}$ is subjected to a finite set of linear homogeneous constraints. By the same argument as that used in the preceding section we then show that KORN'S inequality holds on R.

Thus, we have shown that KORN'S inequality holds in the union of a finite number of domains, each of which can be mapped on a sphere in a twice differentiable non-singular manner.

A sufficient condition for this is that the boundary B is covered by open subsets, on each of which one can introduce a coordinate system with a differentiable metric tensor, and a differentiable vector field whose angle with any inward-pointing vector is bounded away from zero. We map a neighborhood of each open set onto a slab by projecting along the vector field, using the coordinates on B and the distance from B as coordinates. We approximate these slabs by subdomains with smooth boundaries in such a way that the union of these subdomains covers a neighborhood in R of the whole boundary B. We can clearly cover the remainder of R with smooth simply connected regions. Thus, R is the union of a finite set of regions on which KORN'S inequality holds.

Our condition on the vector field permits R to have edges and corners, but not cusps or cusp-like edges.

5. The circle

We consider now the analogous two-dimensional problem of finding the optimum constant K_1 in (1.8) where $D(\vec{u})$ and $S(\vec{u})$ are integrals over the interior of a circle, and i and j take on the values 1, 2. The vector $\vec{u}$ is again subjected to condition (1.4).

In a manner similar to that used in deriving the Korn constant for the sphere, it can be shown that any vector $\vec{u}$ which satisfies (1.4) can be decomposed in the following manner:

$$\vec{u} = \vec{v} + \operatorname{grad}\psi + \vec{w} \tag{5.1}$$

where

$$\operatorname{div}\vec{v} = 0 \tag{5.2}$$

in R and

$$\vec{v} = 0 \tag{5.3}$$

on C. Also $\vec{w}$ is expressible as

$$\vec{w} = 2\left[r^2 \operatorname{grad}\varphi - 2\vec{r}\, r\frac{\partial\varphi}{\partial r}\right] + (r^2 - 1)\operatorname{grad}\left(\varphi + r\frac{\partial\varphi}{\partial r}\right), \tag{5.4}$$

where φ is a harmonic function. It is easily verified that $\vec{v}$, $\operatorname{grad}\psi$ and $\vec{w}$ are mutually orthogonal in both $[D(\vec{u})]^{\frac{1}{2}}$ and $[S(\vec{u})]^{\frac{1}{2}}$ norms.

As in §3 we note that for functions $\vec{v}$ satisfying (5.2) and (5.3)

$$D(\vec{v}) = 2S(\vec{v}). \tag{5.5}$$

Also as before

$$D(\operatorname{grad}\psi) = S(\operatorname{grad}\psi). \tag{5.6}$$

It is easily verified that

$$D(\vec{w}) = 4S(\vec{w}). \tag{5.7}$$

We thus establish the curious fact that the spectrum of $D(\vec{u})$ with respect to $S(\vec{u})$ under condition (1.4) consists of only three eigenvalues ($K = 1, 2, 4$), each occurring with infinite multiplicity. We find the sharp inequality

$$D(\vec{u}) \leqq 4S(\vec{u}), \tag{5.8}$$

so that the best K_1 for the circle is 4. BERNSTEIN & TOUPIN [*1*] showed that $K_1 \leqq 7$.

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Theorems in Linear Elastostatics for Exterior Domains

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1. Introduction

The conventional proofs of fundamental theorems in three-dimensional classical elastostatics, such as the traditional proofs of the uniqueness and reciprocal theorems or of the minimum energy principles[1], are confined in their validity to bounded domains, although this limitation is not always made explicit in the treatise literature. The present paper aims at generalizations of the foregoing theorems to unbounded domains exterior to a finite number of closed surfaces[2].

The generalizations to which we have alluded are elementary in the presence of sufficiently stringent restrictions upon the behavior at infinity of the relevant elastostatic field quantities. Thus Kirchhoff's [*4*] uniqueness proof at once carries over to exterior domains provided the cartesian components of displacement and stress, u_i and τ_{ij}, obey the conditions

$$u_i(x) = c_i + O(r^{-1}), \qquad \tau_{ij}(x) = c_{ij} + O(r^{-2}) \quad \text{as} \quad r \to \infty, \tag{1.1}$$ [3]

where r is the distance from the origin, c_i and c_{ij} ($c_{ji} = c_{ij}$) are prescribed real constants, and the notion of "order of magnitude" is used in its standard mathematical connotation[4].

While requirements (1.1), together with appropriate smoothness hypotheses, are sufficient to assure the uniqueness of the solution to the fundamental boundary-value problems of elastostatics for exterior domains, such *a priori* assumptions as to the characteristic orders of magnitude at infinity are quite artificial: the *rate* at which the displacements and stresses approach their limiting values at infinity is an item of information which one would legitimately expect to infer from the solution to the problem, rather than a condition to be imposed on the solution in advance. By the same token, a uniqueness theorem resting on (1.1) leaves in doubt the status of solutions to exterior boundary-value problems

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[1] See, for example, Love [*1*], Sokolnikoff [*2*], Timoshenko & Goodier [*3*]. Numbers in brackets refer to the list of publications at the end of this paper.

[2] Infinite regions whose boundary extends to infinity are excluded from the following considerations.

[3] The single argument x denotes the triplet of cartesian coordinates x_i. Throughout this paper subscripts have the range of the integers (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma indicate differentiation with respect to the corresponding cartesian coordinate.

[4] See Section 2.

since it does not preclude the existence of alternative solutions to the "same problems", whose displacements and stresses approach their values at infinity less rapidly; nor could such solutions be rejected on convincing physical grounds.

As far as the first boundary-value problem (given surface displacements) is concerned, the preceding uniqueness issue was resolved in a recent paper by DUFFIN & NOLL [5]. In this instance, as shown in [5][5], conditions (1.1) may, without loss in uniqueness, be replaced by the mere requirement that the displacements uniformly tend to given values as $r \to \infty$.

The method of attack employed by DUFFIN & NOLL is well suited to their specific purpose but is not readily adaptable to a similar extension of the uniqueness theorem if the stresses are prescribed at infinity and in case the boundary conditions involve given surface tractions or are of the mixed type. Also, analogous generalizations to exterior domains of other basic theorems in elastostatics would appear to call for a different approach.

In Section 5 of the present paper we prove two theorems concerning the behavior of elastostatic fields at infinity, which enable us to meet the broader objectives just mentioned. A specialization of the first of these theorems yields the conclusion that if the *displacements* tend to zero uniformly at infinity, the displacements and stresses necessarily obey (1.1) with $c_i = c_{ij} = 0$, provided the body forces are suitably restricted. Subject to a similar limitation of the body forces, one infers from the second theorem that the uniform vanishing of the *stresses* at infinity implies that the stresses and displacements, except for an additive rigid displacement field, are again governed by (1.1) with $c_i = c_{ij} = 0$.

The material preceding Section 5 is preliminary to the two theorems cited and is partly expository in character. In Section 2 we collect certain facts pertaining to solid and surface harmonics, and recall a known theorem on the behavior of functions harmonic in a neighborhood of infinity. A corresponding theorem for biharmonic functions is deduced in Section 3. In Section 4 we recall certain properties of elastostatic fields and deal with theorems concerning the representation of such fields in a neighborhood of infinity.

Finally, in Section 6, we extend various basic theorems of classical elastostatics to exterior domains by utilizing the results established in Section 5. Specifically, we give appropriate generalizations of the reciprocal theorem, the principle of work and strain energy, the uniqueness theorem, and the two minimum energy principles. The present extension of the uniqueness theorem assures the uniqueness of the stresses belonging to a sufficiently smooth solution of the general mixed boundary-value problem for an exterior domain if either (a) the displacements or (b) the stresses tend uniformly to pre-assigned constant values at infinity (regardless of the rate at which these limiting values are approached), provided in case (b) the resultant force and moment of the surface tractions on the boundary of the medium are also prescribed. This supplementary condition may, of course, be omitted if the surface tractions are given on the entire boundary. An analogous uniqueness theorem has long been known to hold in two-dimensional elastostatics.

[5] See also [6], which contains a slightly more compact version of the proof presented in [5].

2. Preliminary definitions. Behavior of harmonic functions near infinity

Let (r, ϑ, φ) be spherical coordinates, related to the rectangular cartesian coordinates x_i through the mapping

$$\begin{gathered} x_1 = r \sin\vartheta \cos\varphi, \qquad x_2 = r \sin\vartheta \sin\varphi, \qquad x_3 = r\cos\vartheta, \\ 0 \leqq r < \infty, \qquad 0 \leqq \vartheta \leqq \pi, \qquad 0 \leqq \varphi < 2\pi. \end{gathered} \tag{2.1}$$

A *deleted neighborhood of infinity*, henceforth denoted by $\mathscr{E}$, shall be a region characterized by $r_0 < r < \infty$, and thus bounded internally by a sphere of radius r_0. We note that a function of position defined in $\mathscr{E}$ need not possess a limit as $r \to \infty$.

If $v(r, \vartheta, \varphi)$ is a scalar field defined in $\mathscr{E}$, we write $v(r, \vartheta, \varphi) = O(r^n)$ or $v(r, \vartheta, \varphi) = o(r^n)$ according as $|r^{-n} v(r, \vartheta, \varphi)|$ remains bounded (uniformly in ϑ and φ) or $r^{-n} v(r, \vartheta, \varphi)$ tends to zero (uniformly in ϑ and φ), when $r \to \infty$. A function $v(r, \vartheta, \varphi)$ defined and continuous together with all its partial derivatives of order n $(n \geqq 0)$ in some region of space, will be referred to as being of class $C^{(n)}$ in that region. If $v(r, \vartheta, \varphi)$ is of class $C^{(2)}$ and satisfies Laplace's equation $\nabla^2 v = 0$ in an open region R, it will be called *harmonic* in R; if $v(r, \vartheta, \varphi)$ is of class $C^{(4)}$ and meets $\nabla^4 v = 0$ in R, it will be called *biharmonic* in R.

We now summarize certain well known results in the theory of harmonic functions[6], which will be needed later on. In this connection we recall first that the general *solid spherical harmonic* of degree k admits the representation

$$H^{(k)}(r, \vartheta, \varphi) = S^{(k)}(\vartheta, \varphi)\, r^k \qquad (k = 0, \pm 1, \pm 2, \ldots), \tag{2.2}$$

where $S^{(k)}(\vartheta, \varphi)$ is the general *surface spherical harmonic* of the same degree, and is given by

$$S^{(k)}(\vartheta, \varphi) = \sum_{n=0}^{|k|} [a_k^{(n)} \cos n\varphi + b_k^{(n)} \sin n\varphi]\, P_k^{(n)}(\cos\vartheta), \qquad b_k^{(0)} = 0. \tag{2.3}$$

Here $P_k^{(n)}$ designates the associated Legendre function of the first kind, of degree k and order n, while $a_k^{(0)}$, $a_k^{(n)}$, $b_k^{(n)}$ $(n = 1, 2, \ldots, |k|)$, for fixed k, are $2k+1$ arbitrary constants. If k and n are non-negative integers,

$$P_k^{(n)}(\xi) = (1 - \xi^2)^{\frac{1}{2}n} \frac{d^n P_k(\xi)}{d\xi^n}, \qquad P_k(\xi) = \frac{1}{2^k k!} \frac{d^k (\xi^2 - 1)^k}{d\xi^k}, \tag{2.4}$$

in which P_k is the Legendre polynomial of degree k. Consequently

$$P_k^{(0)}(\xi) = P_k(\xi), \qquad P_k^{(n)}(\xi) = 0 \qquad (n > k \geqq 0). \tag{2.5}$$

In addition we have the recursion relations

$$P_{-k-1}^{(n)}(\xi) = P_k^{(n)}(\xi), \tag{2.6}$$

which are valid for unrestricted k and n. As is evident from (2.3), (2.5), (2.6), *every surface harmonic of degree $-1-k$ is a member of the family of surface harmonics of degree k.* Next we recall that any two surface harmonics of distinct degree are orthogonal over the unit sphere $r = 1$. With the notation

$$[S^{(k)}, S^{(m)}] = \int_0^{2\pi} \int_0^{\pi} S^{(k)}(\vartheta, \varphi)\, S^{(m)}(\vartheta, \varphi) \sin\vartheta\, d\vartheta\, d\varphi \tag{2.7}$$

[6] See, for example, Poincaré [7], Kellogg [8], and Hobson [9].

we thus have

$$[S^{(k)}, S^{(m)}] = 0 \qquad (k \neq m,\ k \neq -m-1). \tag{2.8}$$

The solid harmonics $H^{(k)}(r, \vartheta, \varphi)$ are harmonic functions in $\mathscr{E}$, which are homogeneous of degree k with respect to the cartesian coordinates (x_1, x_2, x_3); they are homogeneous polynomials of degree k in the x_i if k is a non-negative integer. Further, if $H^{(k)}(r, \vartheta, \varphi)$ is a solid harmonic of degree k,

$$H^{(k)}_{,i}(r, \vartheta, \varphi) = \widetilde{H}^{(k-1)}(r, \vartheta, \varphi) \qquad (k = 0, \pm 1, \pm 2, \ldots), \tag{2.9}$$

where $\widetilde{H}^{(k-1)}(r, \vartheta, \varphi)$ is a solid harmonic of degree $k-1$. Also any *specific* solid harmonic of degree k satisfies the identity

$$x_i H^{(k)}_{,i}(r, \vartheta, \varphi) = k H^{(k)}(r, \vartheta, \varphi) \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{2.10}$$

Turning from solid spherical harmonics to general harmonic functions, we cite the following theorem.

Theorem 2.1. *Let $H(r, \vartheta, \varphi)$ be harmonic in $\mathscr{E}$. Then*

(a) *$H(r, \vartheta, \varphi)$ admits the representation*

$$H(r, \vartheta, \varphi) = \sum_{k=-\infty}^{\infty} H^{(k)}(r, \vartheta, \varphi), \tag{2.11}$$

where the $H^{(k)}(r, \vartheta, \varphi)$ are uniquely determined solid spherical harmonics of degree k and the infinite series is uniformly convergent in every closed subregion of $\mathscr{E}$;

(b) *$H(r, \vartheta, \varphi)$ in $\mathscr{E}$ has partial derivatives of all orders, series representations of which may be obtained by performing the corresponding termwise differentiations in* (2.11), *the resulting expansions being also uniformly convergent in every closed subregion of $\mathscr{E}$;*

(c) *if n is a fixed integer, the three statements*

(α) $H(r, \vartheta, \varphi) = O(r^{n-1})$,

(β) $H(r, \vartheta, \varphi) = o(r^n)$,

(γ) $H^{(k)}(r, \vartheta, \varphi) = 0$ *in* (2.11) *for* $k \geqq n$,

are equivalent and imply

(δ) $H_{,i}(r, \vartheta, \varphi) = O(r^{n-2})$.

3. Behavior of biharmonic functions near infinity

In this section we state and prove a theorem which is the counterpart for biharmonic functions of Theorem 2.1.

Theorem 3.1. *Let $F(r, \vartheta, \varphi)$ be biharmonic in $\mathscr{E}$. Then*

(a) *$F(r, \vartheta, \varphi)$ admits the representation*

$$F(r, \vartheta, \varphi) = \sum_{k=-\infty}^{\infty} h^{(k)}(r, \vartheta, \varphi) + r^2 \sum_{k=-\infty}^{\infty} H^{(k)}(r, \vartheta, \varphi), \tag{3.1}$$

where $h^{(k)}(r, \vartheta, \varphi)$ and $H^{(k)}(r, \vartheta, \varphi)$ are solid spherical harmonics of degree k and both infinite series are uniformly convergent in every closed subregion of $\mathscr{E}$;

(b) *$F(r, \vartheta, \varphi)$ in $\mathscr{E}$ has partial derivatives of all orders, series representations of which may be obtained by performing the corresponding termwise differentiations in* (3.1), *the resulting expansions being also uniformly convergent in every closed subregion of $\mathscr{E}$;*

(c) *if n is a fixed integer, the three statements*

(α) $F(r, \vartheta, \varphi) = O(r^{n-1})$,

(β) $F(r, \vartheta, \varphi) = o(r^{n})$,

(γ) $h^{(k)}(r, \vartheta, \varphi) = H^{(k-2)}(r, \vartheta, \varphi) = 0$ *in* (3.1) *for* $k \geqq n$,

are equivalent and imply

(δ) $F_{,i}(r, \vartheta, \varphi) = O(r^{n-2})$.

Proceeding to the proof of the foregoing theorem, we show first that any function $F(r, \vartheta, \varphi)$ which is biharmonic in $\mathscr{E}$, admits ALMANSI's representation

$$F(r, \vartheta, \varphi) = h(r, \vartheta, \varphi) + r^2 H(r, \vartheta, \varphi), \tag{3.2}$$

where $h(r, \vartheta, \varphi)$ and $H(r, \vartheta, \varphi)$ are functions harmonic in $\mathscr{E}$. A completeness proof for the representation (3.2), applicable to a region that is star-shaped with respect to the origin and contains the origin in its interior, is indicated in [*10*] [7]. To establish the completeness of (3.2) in the present circumstances, it is evidently sufficient to exhibit a function $H(r, \vartheta, \varphi)$ which is of class $C^{(2)}$ in $\mathscr{E}$ and there meets

$$\nabla^2 H = 0, \qquad \nabla^2 (F - r^2 H) = 0, \tag{3.3}$$

or, equivalently,

$$\nabla^2 H = 0, \qquad 4r \frac{\partial H}{\partial r} + 6H = \nabla^2 F. \tag{3.4}$$

Since, by hypothesis, $\nabla^2 F$ is harmonic in $\mathscr{E}$, we know from Theorem 2.1 and (2.2) that it admits the expansion

$$\nabla^2 F = \sum_{k=-\infty}^{\infty} \hat{S}^{(k)}(\vartheta, \varphi)\, r^k, \tag{3.5}$$

where the $\hat{S}^{(k)}(\vartheta, \varphi)$ are surface harmonics of degree k and the infinite series has the convergence properties asserted in parts (a), (b) of Theorem 2.1. Now, consider $H(r, \vartheta, \varphi)$ defined by

$$H(r, \vartheta, \varphi) = \sum_{k=-\infty}^{\infty} S^{(k)}(\vartheta, \varphi)\, r^k, \qquad S^{(k)}(\vartheta, \varphi) = \frac{\hat{S}^{(k)}(\vartheta, \varphi)}{2(2k+3)}. \tag{3.6}$$

Clearly, the infinite series in (3.6) has the same convergence properties as that in (3.5). Hence the function $H(r, \vartheta, \varphi)$ given by (3.6) is harmonic in $\mathscr{E}$; further, this function also satisfies the second of (3.4), as is confirmed by direct substitution from (3.6) and by recourse to (3.5).

On applying (a) and (b) of Theorem 2.1 to h and H in (3.2), (a) and (b) of the present theorem follow at once. With a view toward establishing (c), we observe that (β) is immediate from (α) by virtue of the meaning of the symbols "O" and "o". We show next that (β) implies (γ). Let $s^{(k)}(\vartheta, \varphi)$ and $S^{(k)}(\vartheta, \varphi)$ be the respective surface harmonics corresponding to the solid harmonics

[7] See FRANK & v. MISES [*10*], p. 848 *et seq.*

$h^{(k)}(r, \vartheta, \varphi)$ and $H^{(k)}(r, \vartheta, \varphi)$ entering the infinite series in (3.1). We now multiply the right-hand member of (3.1) by $r^{-n} s^{(m)}(\vartheta, \varphi)$ $(m=0, \pm 1, \pm 2, \ldots)$ and integrate termwise over $0 \leqq \vartheta \leqq \pi$, $0 \leqq \varphi \leqq 2\pi$. Bearing in mind (2.2), the convergence properties asserted in (a), as well as the orthogonality relations (2.8), and using the notation (2.7), we find in this manner that the integral

$$I_{mn}(r) \equiv r^{-n} \int_0^{2\pi} \int_0^{\pi} F(r, \vartheta, \varphi)\, s^{(m)}(\vartheta, \varphi) \sin\vartheta \, d\vartheta \, d\varphi, \tag{3.7}$$

for all sufficiently large values of r, is given by

$$\begin{aligned} I_{mn}(r) = r^{m-n}[s^{(m)}, s^{(m)}] + r^{-n-m+1}[s^{(-m-1)}, s^{(m)}] + \\ + r^{m-n+2}[S^{(m)}, s^{(m)}] + r^{-n-m+1}[S^{(-m-1)}, s^{(m)}]. \end{aligned} \tag{3.8}$$

It is apparent from (3.7), because of (β) and since $s^{(m)}(\vartheta, \varphi)$ is bounded on $0 \leqq \vartheta \leqq \pi$, $0 \leqq \varphi \leqq 2\pi$, that $I_{mn}(r)$ tends to zero as $r \to \infty$ for fixed m and n; consequently the same is true of the right-hand member of (3.8). Noting that the coefficients of the inner products appearing in (3.8) are four *distinct* powers of r, we conclude that $[s^{(m)}, s^{(m)}] = 0$ if $m - n \geqq 0$. Recalling that $s^{(m)}(\vartheta, \varphi)$ is real-valued and taking account of (2.2), (2.7), we thus confirm that $s^{(m)}(\vartheta, \varphi) \equiv h^{(m)}(r, \vartheta, \varphi) \equiv 0$ for $m \geqq n$. On multiplying the right-hand member of (3.1) by $r^{-n} S^{(m)}(\vartheta, \varphi)$ and proceeding as before, we verify also that $H^{(m)}(r, \vartheta, \varphi) \equiv 0$ for $m \geqq n-2$. Hence (β) is a sufficient condition for (γ).

We have to show further that (γ) implies (α). This is readily seen to be true by observing that each of the series in (3.1), according to (a) and (b), represents a function harmonic in $\mathscr{E}$, and by invoking part (c) of Theorem 2.1. Finally, (γ) implies (δ) because of (b) of the present theorem and (c) of Theorem 2.1. Theorem 3.1 has thus been proved in its entirety.

4. Representation of elastostatic fields in a neighborhood of infinity

The complete system of fundamental field equations in the linear equilibrium theory of homogeneous isotropic elastic solids, with reference to rectangular cartesian coordinates x_i and in indicial notation[8], takes the form

$$2e_{ij} = u_{i,j} + u_{j,i}, \tag{4.1}$$

$$\tau_{ij} = 2\mu \left[\frac{\sigma}{1-2\sigma} \delta_{ij} e_{pp} + e_{ij} \right], \tag{4.2}$$

$$\tau_{ij,j} + f_i = 0. \tag{4.3}$$

Here u_i, e_{ij}, and τ_{ij} are the cartesian components of displacement, strain, and stress, f_i denotes the components of the body-force density, μ and σ stand for the shear modulus and Poisson's ratio, whereas δ_{ij} designates the Kronecker delta. Equations (4.1), (4.2), and (4.3) represent respectively the displacement-strain relations, the stress-strain law, and the stress equations of equilibrium. Necessary and sufficient for the positive definiteness of the strain-energy density are

$$\mu > 0, \quad -1 < \sigma < \tfrac{1}{2}, \tag{4.4}$$

[8] See Footnote No. 3.

and, provided (4.4) hold, the relations (4.2) may be inverted to yield

$$e_{ij}=\frac{1}{2\mu}\left[\tau_{ij}-\frac{\sigma}{1+\sigma}\,\delta_{ij}\,\tau_{pp}\right]. \tag{4.5}$$

Also, if $u_i(x)$ is of class $C^{(2)}$ in the region at hand, elimination of the stresses and strains among (4.1), (4.2), and (4.3) leads to the displacement equations of equilibrium

$$u_{i,jj}+\frac{1}{1-2\sigma}\,u_{j,ji}+\frac{1}{\mu}\,f_i=0. \tag{4.6}$$

For future convenience we introduce the following definition of a "simple body-force field".

Definition 4.1. *A body force field $f_i(x)$ is simple in an open region R if $f_i(x)$ is of class $C^{(2)}$ in R and*

$$f_{i,i}=0,\qquad \varepsilon_{ijp}\,f_{p,j}=0\quad in\ R. \tag{4.7}$$

In (4.7) ε_{ijp} designates the components of the usual alternator, so that $f_i(x)$ here is required to be both solenoidal and irrotational in R. From (4.7) and the smoothness of $f_i(x)$ follows

$$\nabla^2 f_i=0\quad \text{in } R \tag{4.8}$$

for a body-force field which is simple in R; thus $f_i(x)$, in the present instance, is harmonic in R and accordingly has continuous partial derivatives of all orders in that region. We are now in a position to state concisely the subsequent known theorem of elastostatics.

Theorem 4.1. *Let $u_i(x)$ be of class $C^{(2)}$ in an open region of space R. Let $u_i(x)$, $e_{ij}(x)$, and $\tau_{ij}(x)$ in R satisfy* (4.1), (4.2), (4.3), *subject to* (4.4), *the body-force field $f_i(x)$ being simple in R. Then*

(a) *the functions $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$ have continuous partial derivatives of all orders in R;*

(b) $$\nabla^2 u_{i,i}=0,\qquad \varepsilon_{ijp}\,\nabla^2 u_{p,j}=0; \tag{4.9}$$

(c) $$\nabla^4 u_i=\nabla^4 e_{ij}=\nabla^4\tau_{ij}=0\quad in\ R. \tag{4.10}$$

Accordingly, any solution of the fundamental field equations of classical elastostatics which is of class $C^{(2)}$ and corresponds to simple body-forces, is biharmonic, while the associated dilatation and the rotation vector are harmonic in the region under consideration. Part (a) of the theorem just cited was proved by Duffin [*11*] for the special case in which the body-force field vanishes identically; the argument indicated in [*11*] is, however, readily generalized to accommodate any simple body-force field. Once (a) has been established, (b) and (c) follow[9] trivially with the aid of (4.6), (4.7), (4.8), (4.1), and (4.2).

We turn next to two representation theorems for solutions of the elastostatic field equations in a neighborhood of infinity. The first of these theorems concerns the general solution of (4.6) in the absence of body forces; the second aims at a particular solution of (4.6) corresponding to arbitrary simple body forces.

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[9] See, for example, [*2*], p. 77.

Theorem 4.2. *Let $u_i(x)$ be of class $C^{(2)}$ in $\mathscr{E}$ and there satisfy* (4.6) *with $f_i(x)=0$, the elastic constants being subject to* (4.4). *Then*[10] $u_i(r,\vartheta,\varphi)$ *admits the representation*

$$u_i(r,\vartheta,\varphi)=\sum_{k=-\infty}^{\infty} h_i^{(k)}(r,\vartheta,\varphi)+r^2\sum_{k=-\infty}^{\infty} H_i^{(k)}(r,\vartheta,\varphi) \tag{4.11}$$

where $h_i^{(k)}(r,\vartheta,\varphi)$ and $H_i^{(k)}(r,\vartheta,\varphi)$ are solid spherical harmonics of degree k and

$$H_i^{(k)}=\frac{-1}{2[k+1+(1-2\sigma)(2k+3)]}\,h_{j,ji}^{(k+2)} \qquad (k=0,\pm 1,\pm 2,\ldots). \tag{4.12}$$

Both infinite series in (4.11), *as well as the series resulting from any finite number of termwise differentiations of* (4.11), *are uniformly convergent in every closed subregion of $\mathscr{E}$.*

A somewhat less explicit version of this theorem was established by KELVIN [*12*] in connection with his work on elastostatic boundary-value problems for a spherical shell. The following proof, however, would appear to be more direct.

By hypothesis and Theorem 4.1, $u_i(r,\vartheta,\varphi)$ is biharmonic in $\mathscr{E}$. Theorem (3.1) therefore assures the existence of an expansion in the form (4.11) which has the asserted convergence properties.

This leaves only (4.12) yet to be confirmed. From (4.11), (2.10), and the harmonicity of $h^{(k)}$, $H^{(k)}$ follows

$$\nabla^2 u_i\equiv u_{i,jj}=2\sum_{k=-\infty}^{\infty}(2k+3)\,H_i^{(k)}. \tag{4.13}$$

Substitution from (4.13) into (4.9), in turn, leads to

$$\sum_{k=-\infty}^{\infty}(2k+3)\,H_{i,i}^{(k)}=0, \qquad \sum_{k=-\infty}^{\infty}(2k+3)\,\varepsilon_{ijp}H_{p,j}^{(k)}=0 \quad \text{in } \mathscr{E}. \tag{4.14}$$

But $H_{i,j}^{(k)}$, according to (2.9), for fixed i and j, is a solid harmonic of degree $k-1$. Hence (4.14), by virtue of the uniqueness of the expansion (2.11) in Theorem 2.1, imply

$$H_{i,i}^{(k)}=0, \qquad H_{i,j}^{(k)}=H_{j,i}^{(k)} \qquad (k=0\ \pm 1,\pm 2,\ldots). \tag{4.15}$$

With the aid of (4.11), (2.10), and (4.15), we obtain

$$u_{j,ji}=\sum_{k=-\infty}^{\infty}\left[h_{j,ji}^{(k)}+2(k+1)\,H_i^{(k)}\right], \tag{4.16}$$

and inserting (4.13), (4.16) in (4.6) with $f_i=0$, there results

$$\sum_{k=-\infty}^{\infty}\left\{h_{j,ji}^{(k)}+2[k+1+(1-2\sigma)(2k+3)]\,H_i^{(k)}\right\}=0. \tag{4.17}$$

Since $h_{j,ji}^{(k)}$ is a solid harmonic of degree $k-2$, (4.12) is a consequence[11] of (4.17) and Theorem 2.1. The proof of Theorem 4.2 is now complete. Expansions

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[10] If $g(x)\equiv g(x_1,x_2,x_3)$ is a function of the cartesian coordinates x_i, we write consistently $g(r,\vartheta,\varphi)$ in place of $g[x(r,\vartheta,\varphi)]$. This ambiguity in the use of the functional symbol g ought not to cause confusion.

[11] Note that the coefficient of $H_i^{(k)}$ in (4.17) cannot vanish because of (4.4).

for the components of stress and strain associated with the displacement field $u_i(x)$ in Theorem 4.2 may be deduced from (4.11) by means of (4.1) and (4.2).

Theorem 4.3. *Let $f_i(x)$ be a body-force field simple in $\mathscr{E}$. Then*

(a) *$f_i(r, \vartheta, \varphi)$ admits the representation*

$$f_i(r, \vartheta, \varphi) = \sum_{k=-\infty}^{\infty} f_i^{(k)}(r, \vartheta, \varphi), \tag{4.18}$$

where the $f_i^{(k)}(r, \vartheta, \varphi)$ are uniquely determined solid spherical harmonics of degree k,

$$f_{i,i}^{(k)} = 0, \qquad f_{i,j}^{(k)} = f_{j,i}^{(k)} \qquad (k = 0, \pm 1, \pm 2, \ldots), \tag{4.19}$$

and the infinite series (4.18), *as well as the series resulting from any finite number of termwise differentiations of* (4.18), *are uniformly convergent in every closed subregion of $\mathscr{E}$;*

(b) *the same convergence properties apply to the series*

$$u_i(r, \vartheta, \varphi) = -\frac{r^2}{2\mu} \sum_{k=-\infty}^{\infty} \frac{1-2\sigma}{k+1+(1-2\sigma)(2k+3)} f_i^{(k)}(r, \vartheta, \varphi), \tag{4.20}$$

provided the constants μ, σ obey (4.4), *and further the functions u_i so defined satisfy* (4.6) *in $\mathscr{E}$;*

(c) *if n is a fixed integer,*

$$f_i(r, \vartheta, \varphi) = o(r^n) \quad \textit{implies} \quad u_i(r, \vartheta, \varphi) = O(r^{n+1}).$$

Part (a) of this theorem is readily inferred from the simplicity of $f_i(x)$ in $\mathscr{E}$, Definition 4.1, (4.8), and Theorem 2.1.

The uniform convergence of the infinite series (4.20), and of the corresponding series resulting from successive differentiations of (4.20), is apparent from the analogous convergence properties possessed by the series (4.18). Consequently $u_i(r, \vartheta, \varphi)$ has partial derivatives of all orders in $\mathscr{E}$. Substituting from (4.20) into (4.6), carrying out the required differentiations termwise, and using (2.10), (4.18), (4.19), one readily verifies that $u_i(r, \vartheta, \varphi)$ is indeed a particular solution of (4.6). This establishes part (b) of the theorem.

To justify (c) we note from (a) of the present theorem and (c) of Theorem 2.1 that $f_i(r, \vartheta, \varphi) = o(r^n)$ requires $f_i^{(k)}(r, \vartheta, \varphi) = 0$ for $k \geqq n$. Bearing in mind (4.20) and appealing once more to (c) of Theorem 2.1, we see that (c) of the present theorem holds true, and the proof is complete.

5. Behavior of elastostatic fields at infinity

At this stage we may turn to our main objective and examine the behavior at infinity of elastostatic fields which conform to the fundamental field equations in a neighborhood of infinity.

Theorem 5.1. *Let $u_i(x)$ be of class $C^{(2)}$ in $\mathscr{E}$. Suppose $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$ in $\mathscr{E}$ satisfy* (4.1), (4.2), (4.3), *subject to* (4.4), *the body forces being simple in $\mathscr{E}$. Then, if n is a fixed integer,*

$$u_i(x) = o(r^n) \tag{5.1}$$

implies

$$u_i(x) = O(r^{n-1}), \quad e_{ij}(x) = O(r^{n-2}),$$
$$\tau_{ij}(x) = O(r^{n-2}), \quad f_i(x) = O(r^{n-3}). \tag{5.2}$$

By hypothesis and Theorem 4.1, $u_i(x)$ is biharmonic in $\mathscr{E}$. On applying Theorem 3.1 to $u_i(x)$, we draw from (α), (β) in part (c) of this theorem that (5.1) implies the first of (5.2). The remaining equations (5.2) then follow immediately from (4.1), (4.2), (4.3), and (δ) in part (c) of Theorem 3.1. This concludes the proof of the present theorem.

Taking $n=0$ in Theorem 5.1, we find that *if the displacements vanish uniformly at infinity, they must vanish to the order $O(r^{-1})$, while the associated strains and stresses must decay to the order $O(r^{-2})$, and the body forces are necessarily of the order $O(r^{-3})$ at infinity.* It follows, in particular, that the dilatation and the rotation are both $O(r^{-2})$ in the present circumstances — a conclusion which DUFFIN & NOLL [5] reached by entirely different means.

Theorem 5.2. *Let $u_i(x)$ be of class $C^{(2)}$ in $\mathscr{E}$. Suppose $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$ in $\mathscr{E}$ satisfy* (4.1), (4.2), (4.3), *subject to* (4.4), *the body forces being simple in $\mathscr{E}$, with*

$$f_i(x) = o(r^{-2}). \tag{5.3}$$

Then

$$\tau_{ij}(x) = o(1) \tag{5.4}$$

implies

$$u_i(x) = \mathring{u}_i(x) + O(r^{-1}), \quad e_{ij}(x) = O(r^{-2}),$$
$$\tau_{ij}(x) = O(r^{-2}), \quad f_i(x) = O(r^{-3}), \tag{5.5}$$

provided $\mathring{u}_i(x)$ represents an arbitrary rigid displacement field and hence has the form

$$\mathring{u}_i(x) = \mathring{\lambda}_i + \varepsilon_{ijp}\mathring{\omega}_j x_p \qquad (\mathring{\lambda}_i, \mathring{\omega}_i \ldots \mathit{constant}). \tag{5.6}$$

In view of the assumed simplicity of $f_i(x)$ and because of (5.3), we infer from Theorem 4.3 the existence of a particular solution $u_i(x)$ to (4.6) which is of class $C^{(2)}$ in $\mathscr{E}$ and conforms to $u_i(x) = O(r^{-1})$. Accordingly, by virtue of Theorem 5.1, the strains and stresses associated with this particular $u_i(x)$ in the sense of (4.1), (4.2), obey the second and third of (5.5), while $f_i(x)$ must conform to the last of (5.5). We are thus able to exhibit a particular solution to (4.1), (4.2), (4.3) which has the required smoothness and possesses the orders of magnitude (5.5) at infinity. Therefore and since the system (4.1), (4.2), (4.3) is linear, it remains to be demonstrated merely that Theorem 5.2 is true when $f_i(x) = 0$ in $\mathscr{E}$.

Suppose now $f_i(x) = 0$ in $\mathscr{E}$. Then, by hypothesis and Theorem 4.2, $u_i(x)$ admits the representation

$$u_i(x) = \mathring{u}_i(x) + \hat{u}_i(x), \tag{5.7}$$

where $\mathring{u}_i$ and $\hat{u}_i$ are *defined* by

$$\mathring{u}_i(r,\vartheta,\varphi) = \sum_{k=0}^{\infty} h_i^{(k)}(r,\vartheta,\varphi) + r^2 \sum_{k=-2}^{\infty} H_i^{(k)}(r,\vartheta,\varphi),$$
$$\hat{u}_i(r,\vartheta,\varphi) = \sum_{k=-\infty}^{-1} h_i^{(k)}(r,\vartheta,\varphi) + r^2 \sum_{k=-\infty}^{-3} H_i^{(k)}(r,\vartheta,\varphi), \tag{5.8}$$

and the solid harmonics $h_i^{(k)}$, $H_i^{(k)}$ satisfy (4.12). Further, each of the infinite series in (5.8) is uniformly convergent in every closed subregion of $\mathscr{E}$ and may be differentiated termwise arbitrarily often without loss of convergence.

Next, define functions $\mathring{e}_{ij}$ and $\hat{e}_{ij}$ through

$$2\mathring{e}_{ij}(x) = \mathring{u}_{i,j}(x) + \mathring{u}_{j,i}(x),$$
$$2\hat{e}_{ij}(x) = \hat{u}_{i,j}(x) + \hat{u}_{j,i}(x). \tag{5.9}$$

From (5.7), (5.9), and (4.1) we have

$$e_{ij}(x) = \mathring{e}_{ij}(x) + \hat{e}_{ij}(x). \tag{5.10}$$

The two infinite series entering the second of (5.8) each represent a function harmonic in $\mathscr{E}$. The second of (5.9) and part (c) of Theorem 2.1 therefore entitle us to write

$$\hat{u}_i(x) = O(r^{-1}), \qquad \hat{e}_{ij}(x) = O(r^{-2}). \tag{5.11}$$

Because of (4.2), the order of magnitude at infinity of the stresses $\tau_{ij}(x)$ must be the same as that of the strains $e_{ij}(x)$. Also, if $\mathring{e}_{ij}(x)$ vanishes identically, it follows that $\mathring{u}_i(x)$ must have the form (5.6) appropriate to an arbitrary rigid displacement field. Consequently, and by virtue of (5.11), (5.10), (5.7), (4.2), all we have left to show is that $\mathring{e}_{ij}(x) \equiv 0$ in the present circumstances.

To this end we first conclude from (4.12) that

$$H_{i,i}^{(k)} = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{5.12}$$

and hence reach with the aid of (5.8), (5.9),

$$\mathring{e}_{ii} = \sum_{k\,0}^{\infty} h_{i,i}^{(k)} + 2 \sum_{k--2}^{\infty} x_i H_i^{(k)}. \tag{5.13}$$

Eliminating $H_i^{(k)}$ from (5.13) by recourse to (4.12), and at the same time observing on the basis of (2.9), (2.10), (2.3), (2.2) that

$$x_i h_{j,ji}^{(k+2)} = (k+1)\, h_{j,j}^{(k+2)}, \qquad h_{i,j}^{(0)} = 0, \tag{5.14}$$

we arrive at

$$\mathring{e}_{ii} = \sum_{k=1}^{\infty} \frac{(1-2\sigma)(2k-1)}{k-1+(1-2\sigma)(2k-1)}\, h_{i,i}^{(k)}. \tag{5.15}$$

Now, by (5.4) and (4.5),

$$e_{ij}(x) = o(1), \tag{5.16}$$

so that from (5.10) and the second of (5.11)

$$\mathring{e}_{ij}(x) = o(1). \tag{5.17}$$

On applying Theorem 2.1 to the harmonic function $\mathring{e}_{ii}(x)$ given by (5.15), and bearing in mind (2.9), we draw from (5.17) that

$$h_{i,i}^{(k)} = 0 \quad \text{for} \quad k \geqq 1, \tag{5.18}$$

which, together with (4.12) and the second of (5.14), yields

$$H_i^{(k)} = 0 \quad \text{for} \quad k \geqq -2. \tag{5.19}$$

Inserting (5.19) in the first of (5.8), and using the first of (5.9) and the second of (5.14), we obtain

$$2\,\mathring{e}_{ij} = \sum_{k=1}^{\infty} [h^{(k)}_{i,j} + h^{(k)}_{j,i}]. \tag{5.20}$$

Recalling (2.9) again, we gather from (5.20) that $\mathring{e}_{ij}(x)$ is harmonic in $\mathscr{E}$. Also, in view of part (c) of Theorem 2.1, (5.20) is consistent with (5.17) only if

$$\mathring{e}_{ij}(x) = 0 \quad \text{in } \mathscr{E}, \tag{5.21}$$

and the proof of the present theorem is complete.

6. Extension of basic theorems in elastostatics to exterior domains

We now apply the results established in Section 5 to the generalization for infinite regions of fundamental theorems in the linear equilibrium theory of homogeneous and isotropic elastic solids. In this connection we assume that the region occupied by the solid, whether finite or infinite, is a *regular region of space, i.e.* a closed region $D+B$ whose boundary B consists of a finite number of non-intersecting closed "regular surfaces", the latter term being used in the sense of KELLOGG [*8*]. Accordingly, the boundary B of a regular region of space $D+B$ may exhibit corners and edges. Further, D need not be simply connected. If D is infinite, B remains bounded, so that D in this instance is necessarily an exterior domain [12]. We turn first to an extension of Betti's reciprocal theorem.

Theorem 6.1. *Let $D+B$ be a regular region of space. Let $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$, $f_i(x)$ and $u'_i(x)$, $e'_{ij}(x)$, $\tau'_{ij}(x)$, $f'_i(x)$ be two sets of displacement, strain, stress, and body-force fields having the following properties:*

(a) *$u_i(x)$ and $u'_i(x)$ are of class $C^{(2)}$ in $D+B$;*

(b) *each of the two sets of fields in D satisfies* (4.1), (4.2), (4.3) *for the same elastic constants μ and σ, the latter obeying* (4.4);

(c) *if D is infinite, $f_i(x)$ and $f'_i(x)$ are simple in D, in the sense of Definition* 4.1, *and either*

$$u_i(x) = o(1), \qquad u'_i(x) = o(1) \tag{6.1}$$

or

$$\begin{aligned} \tau_{ij}(x) &= o(1), & \tau'_{ij}(x) &= o(1), \\ f_i(x) &= o(r^{-2}), & f'_i(x) &= o(r^{-2}), \end{aligned} \tag{6.2}$$

and

$$\int_B t_i\,dA + \int_D f_i\,dV = 0, \qquad \int_B t'_i\,dA + \int_D f'_i\,dV = 0, \tag{6.3}$$

$$\begin{aligned} \int_B \varepsilon_{ijp}\,x_j\,t_p\,dA + \int_D \varepsilon_{ijp}\,x_j\,f_p\,dV = 0, \\ \int_B \varepsilon_{ijp}\,x_j\,t'_p\,dA + \int_D \varepsilon_{ijp}\,x_j\,f'_p\,dV = 0, \end{aligned} \tag{6.4}$$

where

$$t_i = \tau_{ij}\,n_j, \qquad t'_i = \tau'_{ij}\,n_j, \tag{6.5}$$

while n_j are the components of the outward unit normal to B.

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[12] *Cf.* Footnote No. 2.

Then

$$\int_B t_i u_i' \, dA + \int_D f_i u_i' \, dV = \int_B t_i' u_i \, dA + \int_D f_i' u_i \, dV = \int_D \tau_{ij} e_{ij}' \, dV = \int_D \tau_{ij}' e_{ij} \, dV. \quad (6.6)$$

For the case in which D is finite, the proof of this theorem is well known [13]. Suppose, therefore, D is infinite. Let $R(\varrho)$ be the region whose boundary is $B + \Omega(\varrho)$, where $\Omega(\varrho)$ is a sphere of radius ϱ, centered at the origin, and containing B wholly in its interior. On applying (6.6) to the finite regular region $R(\varrho) + B + \Omega(\varrho)$, we have

$$\begin{aligned} &\int_B t_i u_i' \, dA + \int_{\Omega(\varrho)} t_i u_i' \, dA + \int_{R(\varrho)} f_i u_i' \, dV = \int_{R(\varrho)} \tau_{ij} e_{ij}' \, dV, \\ &\int_B t_i' u_i \, dA + \int_{\Omega(\varrho)} t_i' u_i \, dA + \int_{R(\varrho)} f_i' u_i \, dV = \int_{R(\varrho)} \tau_{ij}' e_{ij} \, dV, \end{aligned} \quad (6.7)$$

$$\int_{R(\varrho)} \tau_{ij}' e_{ij} \, dV = \int_{R(\varrho)} \tau_{ij} e_{ij}' \, dV. \quad (6.8)$$

Assuming first (6.1) to hold, we draw from Theorem 5.1 (with $n=0$) that

$$u_i(x) = O(r^{-1}), \quad u_i'(x) = O(r^{-1}), \quad (6.9)$$

$$\begin{aligned} e_{ij}(x) &= O(r^{-2}), \quad \tau_{ij}(x) = O(r^{-2}), \quad f_i(x) = O(r^{-3}), \\ e_{ij}'(x) &= O(r^{-2}), \quad \tau_{ij}'(x) = O(r^{-2}), \quad f_i'(x) = O(r^{-3}). \end{aligned} \quad (6.10)$$

According to (6.9), (6.10), and (6.5),

$$\int_{\Omega(\varrho)} t_i u_i' \, dA \to 0, \quad \int_{\Omega(\varrho)} t_i' u_i \, dA \to 0 \quad \text{as} \quad \varrho \to \infty, \quad (6.11)$$

while the volume integrals over $R(\varrho)$ in (6.7), (6.8) are convergent in this limit and tend to the corresponding (improper) integrals over D. Hence (6.6) follow by passing to the limit as $\varrho \to \infty$ in (6.7), (6.8).

Next, assume that (6.2), (6.3), (6.4) hold instead of (6.1). In connection with this alternative hypothesis, it should be noted that (6.3), (6.4) are a consequence of the stress equations of equilibrium (4.3) and of the symmetry of the stress tensor if D is bounded, but constitute an independent assumption in the present circumstances; conditions (6.3), (6.4) could, however, be replaced with the requirement that, as $\varrho \to \infty$,

$$\begin{aligned} &\int_{\Omega(\varrho)} t_i \, dA \to 0, \qquad \int_{\Omega(\varrho)} t_i' \, dA \to 0, \\ &\int_{\Omega(\varrho)} \varepsilon_{ijp} x_j t_p \, dA \to 0, \quad \int_{\Omega(\varrho)} \varepsilon_{ijp} x_j t_p' \, dA \to 0. \end{aligned} \quad (6.12)$$

In any event, Theorem 5.2 implies that

$$u_i(x) = \mathring{u}_i(x) + \hat{u}_i(x), \quad u_i'(x) = \mathring{u}_i' + \hat{u}_i'(x), \quad (6.13)$$

in which $\mathring{u}_i(x)$ and $\mathring{u}_i'(x)$ are rigid displacement fields and

$$\hat{u}_i(x) = O(r^{-1}), \quad \hat{u}_i'(x) = O(r^{-1}); \quad (6.14)$$

further, (6.10) continue to hold true.

[13] See, for example, [2], art. 109.

Since $\hat{u}_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$, $f_i(x)$ and $\hat{u}'_i(x)$, $e'_{ij}(x)$, $\tau'_{ij}(x)$, $f'_i(x)$ have properties (a), (b), and conform to (6.1), it follows from what has already been shown that

$$\int_B t_i \hat{u}'_i \, dA + \int_D f_i \hat{u}'_i \, dV = \int_B t'_i \hat{u}_i \, dA + \int_D f'_i \hat{u}_i \, dV = \int_D \tau_{ij} e'_{ij} \, dV = \int_D \tau'_{ij} e_{ij} \, dV. \quad (6.15)$$

On the other hand, bearing in mind that

$$\mathring{u}_i(x) = \mathring{\lambda}_i + \varepsilon_{ijp} \mathring{\omega}_j x_p, \quad \mathring{u}'_i(x) = \mathring{\lambda}'_i + \varepsilon_{ijp} \mathring{\omega}'_j x_p, \quad (6.16)$$

where $\mathring{\lambda}_i$, $\mathring{\omega}_i$ and $\mathring{\lambda}'_i$, $\mathring{\omega}'_i$ are constants, we conclude from (6.3), (6.4) that

$$\int_B t_i \mathring{u}'_i \, dA + \int_D f_i \mathring{u}'_i \, dV = 0, \quad \int_B t'_i \mathring{u}_i \, dA + \int_D f'_i \mathring{u}_i \, dV = 0. \quad (6.17)$$

Equations (6.15), (6.17), because of (6.13), imply (6.6). This completes the proof of the theorem. Theorem 6.1 at once yields, as a special case, the following analogous extension of the principle of work and strain energy [14].

Theorem 6.2. *Let $D+B$, $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$, $f_i(x)$, as well as μ and σ, satisfy the same hypotheses as in* Theorem 6.1. *Then*

$$\int_B t_i u_i \, dA + \int_D f_i u_i \, dV - 2 \int_D W \, dV, \quad (6.18)$$

where $W(x)$ is the strain-energy density given by

$$W = \frac{1}{2} \tau_{ij} e_{ij} = \mu \left[\frac{\sigma}{1-2\sigma} e_{ii} e_{jj} + e_{ij} e_{ij} \right], \quad (6.19)$$

and t_i are the components of the surface traction on B, given by the first of (6.5).

The truth of this theorem is immediately confirmed by setting $u_i(x) = u'_i(x)$ in Theorem 6.1 and by invoking (4.2). Theorem 6.2 serves as a basis for the subsequent generalized version of the uniqueness theorem of elastostatics.

Theorem 6.3. *Let $D+B$ be a regular region of space. Let $u'_i(x)$, $e'_{ij}(x)$, $\tau'_{ij}(x)$ and $u''_i(x)$, $e''_{ij}(x)$, $\tau''_{ij}(x)$ be two sets of displacement, strain, and stress fields having the following properties:*

(a) *$u'_i(x)$ and $u''_i(x)$ are of class $C^{(2)}$ in $D+B$;*

(b) *each of the two sets of fields in D satisfies* (4.1), (4.2), (4.3), *for the same body-force field $f_i(x)$ and the same elastic constants μ and σ, the latter obeying* (4.4);

(c)
$$\begin{aligned} u'_i(x) &= u''_i(x) && \text{on } B_1, \\ t'_i(x) \equiv \tau'_{ij} n_j &= \tau''_{ij} n_j \equiv t''_i(x) && \text{on } B_2, \end{aligned} \quad (6.20)$$

where B_1 and B_2 are complementary subsets of B, n_j denotes the components of the outward unit normal of B, while t'_i and t''_i are the components of the respective surface tractions;

(d) *if D is infinite, either*

$$u'_i(x) = c_i + o(1), \quad u''_i(x) = c_i + o(1) \quad (c_i \ldots \textit{constant}), \quad (6.21)$$

or

$$\tau'_{ij}(x) = c_{ij} + o(1), \quad \tau''_{ij}(x) = c_{ij} + o(1) \quad (c_{ij} \ldots \textit{constant}), \quad (6.22)$$

[14] See, for example, [2], art. 27.

with $c_{ji}=c_{ij}$, *and*

$$\int_B t'_i\,dA=\int_B t''_i\,dA, \tag{6.23}$$

$$\int_B \varepsilon_{ijp}\,x_j\,t'_p\,dA=\int_B \varepsilon_{ijp}\,x_j\,t''_p\,dA, \tag{6.24}$$

i.e., *the two systems of surface tractions on B are statically equivalent.*

Then

$$u'_i(x)=u''_i(x)+\mathring{u}_i(x),\quad e'_{ij}(x)=e''_{ij}(x),\quad \tau'_{ij}(x)=\tau''_{ij}(x)\quad in\quad D+B, \tag{6.25}$$

provided $\mathring{u}_i(x)$ *designates a rigid displacement field.*

We emphasize that Theorem 6.3, in contrast to the preceding two theorems, involves no explicit restrictions of the body-force field. A proof of this theorem is readily carried out along the lines of the traditional uniqueness argument appropriate to finite domains[15]. Thus let

$$\begin{gathered} u_i(x)=u'_i(x)-u''_i(x),\quad e_{ij}(x)=e'_{ij}(x)-e''_{ij}(x),\\ \tau_{ij}(x)=\tau'_{ij}(x)-\tau''_{ij}(x). \end{gathered} \tag{6.26}$$

Clearly, $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$ meet the hypotheses of Theorem 6.2 with $f_i(x)=0$. Therefore, and by virtue of (6.20), (6.26),

$$\int_D W\,dV=0, \tag{6.27}$$

where $W(x)$ is the strain-energy density associated with $e_{ij}(x)$, and the integral in (6.27) is improper but convergent if D is infinite. The second of (6.25) is a consequence of (6.26), (6.27), and the fact that W, in accordance with (6.19) and (4.4), is a positive definite quadratic form in the e_{ij}. The remaining equations in (6.25) then follow from (4.1), (4.2), and this completes the proof. Evidently, $\mathring{u}_i(x)$ in (6.25) vanishes identically if B_1 contains at least one subregion of B with non-vanishing area or — in the event that D is infinite — if the regularity conditions at infinity take the form of (6.21).

When D is infinite, B_2 is empty, and (6.21) are required to hold, Theorem 6.3 yields in particular the uniqueness theorem for the first boundary-value problem which Duffin & Noll [*5*][16] established by different means, except that their argument remains valid even if μ and σ merely meet

$$\mu\neq 0,\quad -\infty<\sigma<\tfrac{1}{2},\quad 1<\sigma<\infty, \tag{6.28}$$

rather than the more restrictive assumptions[17] (4.4). A uniqueness theorem analogous to Theorem 6.3 is known to apply to two-dimensional elastostatics[18]. We also draw attention to the fact that the extension of the present theorem to mixed-mixed boundary-value problems is entirely elementary.

If D is infinite and B_1 is empty, *i.e.*, if the surface tractions are prescribed over the entire boundary (second boundary-value problem) of a medium occupying

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[15] See, for example, [*2*], art. 27.

[16] See also [*6*].

[17] Theorem 6.3, of course, continues to hold if $\mu>0$ in (4.4) is replaced by $\mu<0$, but the inequalities on σ in (4.4) are essential to the argument we have used.

[18] See Muskhelisvili [*13*], art. 40.

an infinite regular region of space, conditions (6.23), (6.24) in Theorem 6.3 may be omitted because they are implied by (6.20). On the other hand, if B_1 is not empty (first or mixed boundary-value problem), (6.23), (6.24) assume the status of independent additional hypotheses. Further, in this instance the two hypotheses under consideration, which require the prescription of the resultant force and moment (about the origin) of the surface tractions on B, appear at first sight artificial since the tractions over at least a portion of the boundary are here not known beforehand. One is therefore led to ask whether (6.23), (6.24) are *necessary* for the truth of the uniqueness Theorem 6.3 when D is infinite and the regularity conditions at infinity are taken in the form (6.22). That this is indeed the case is clear from the next theorem.

Theorem 6.4. *Theorem 6.3 is false if either the hypothesis (6.23) or the hypothesis (6.24) is omitted.*

For the purpose at hand, let $D+B$ be the region exterior to the unit sphere about the origin, assume that $B=B_1$ (B_2 empty) in Theorem 6.3, take $c_{ij}=0$, and suppose that $u_i''(x)$, $e_{ij}''(x)$, $\tau_{ij}''(x)$, and $f_i(x)$ vanish identically in $D+B$. Conditions (6.23), (6.24) then become

$$\int_B t_i' \, dA = 0, \tag{6.29}$$

$$\int_B \varepsilon_{ijp} x_j t_p' \, dA = 0. \tag{6.30}$$

To prove Theorem 6.4, it clearly suffices to show that if either (6.29) or (6.30) are violated, there exist fields $u_i'(x)$, $e_{ij}'(x)$, $\tau_{ij}'(x)$, which conform to hypotheses (a) and (b) of Theorem 6.3 with $f_i(x)=0$, and are such that the strains $e_{ij}'(x)$ do not vanish identically in $D+B$, while

$$u_i'(x)=0 \quad \text{on} \quad B(x_i x_i=1), \tag{6.31}$$

$$\tau_{ij}'(x)=o(1). \tag{6.32}$$

We now demonstrate with the aid of two examples [19] that such fields do in fact exist.

To this end consider the displacement field

$$u_i'=\frac{\lambda_i}{r}+\frac{r^2-1}{2(5-6\sigma)}\left(\frac{\lambda_j}{r}\right)_{,ij}-\lambda_i \qquad (\lambda_i \neq 0), \tag{6.33}$$

in which the λ_i are constants, and the displacement field

$$u_i'=\frac{\varepsilon_{ipq}\,\omega_p\,x_q}{r^3}+\frac{r^2-1}{4(4-5\sigma)}\left(\frac{\varepsilon_{jpq}\,\omega_p\,x_q}{r^3}\right)_{,ij}-\varepsilon_{ipq}\,\omega_p\,x_q \qquad (\omega_i \neq 0), \tag{6.34}$$

in which the ω_i are constants. Both of the foregoing displacement fields, together with their associated fields of strain and stress, possess all of the required properties, as is readily confirmed. The leading two terms in (6.33) represent the displacements induced in the medium by a rigid translation (with components λ_i) applied to the spherical boundary B, if the body is constrained against displacements at infinity; the last term in (6.33) corresponds to a rigid translation (with

[19] These examples were suggested by Professor R. T. SHIELD.

components $-\lambda_i$) of the entire medium. On the other hand, the leading two terms in (6.34) represent the displacements generated in the medium by an infinitesimal rotation (with components ω_i) applied to B, if the body is held fixed at infinity, while the last term in (6.34) corresponds to an infinitesimal rotation (with components $-\omega_i$) of the entire medium. Furthermore, if L'_i and M'_i respectively denote the components of the resultant force and of the resultant moment about the origin of the surface tractions on the unit sphere B, we find from (6.33) that

$$L'_i = 24\pi\mu \frac{1-\sigma}{5-6\sigma} \lambda_i \neq 0, \qquad M'_i = 0, \tag{6.35}$$

whereas the displacement field (6.34) gives rise to

$$L'_i = 0, \qquad M'_i = 8\pi\mu\omega_i \neq 0. \tag{6.36}$$

Thus, the solution of the field equations characterized by (6.33) violates (6.29), whereas that characterized by (6.34) fails to meet (6.30). The proof of Theorem 6.4 is now complete.

Our next objective is a generalization to infinite domains of the principle of minimum potential energy. For this purpose it is helpful to introduce the subsequent definition.

Definition 6.1. *Let $\tilde{u}_i(x)$ be a displacement field with the following properties:*

(a) *$\tilde{u}_i(x)$ is of class $C^{(2)}$ in a regular region of space $D+B$;*

(b) $$\tilde{u}_i = u_i^*(x) \quad \text{on } B_1, \tag{6.37}$$

where B_1 is a subset of B and $u_i^*(x)$ is a triplet of functions prescribed on B_1;

(c) *if D is infinite,*

$$\tilde{u}_i(x) = O(r^{-1}), \qquad \tilde{e}_{ij}(x) = O(r^{-2}), \tag{6.38}$$

where $\tilde{e}_{ij}(x)$ are the strains associated with $\tilde{u}_i(x)$ in the sense of (4.1).

The totality of all such $\tilde{u}_i(x)$ constitutes the class of kinematically admissible displacement fields $[\tilde{u}]$ corresponding to $D+B$, B_1, and $u_i^*(x)$.

Theorem 6.5. *Let $D+B$ be a regular region of space. Let $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$, $f_i(x)$ be a set of displacement, strain, stress, and body-force fields having the following properties:*

(a) *$u_i(x)$ is of class $C^{(2)}$ in $D+B$;*

(b) *$u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$, and $f_i(x)$ in D satisfy* (4.1), (4.2), (4.3) *for elastic constants obeying* (4.4);

(c) $$u_i = u_i^*(x) \quad \text{on} \quad B_1, \qquad \tau_{ij} n_j = t_i^*(x) \quad \text{on } B_2, \tag{6.39}$$

where B_1, B_2, and n_j have the same meaning as in Theorem 6.3, *while $u_i^*(x)$ and $t_i^*(x)$ are functions prescribed on B_1 and B_2, respectively;*

(d) *if D is infinite, $f_i(x)$ is simple in D, in the sense of Definition* 4.1, *and*

$$u_i(x) = o(1). \tag{6.40}$$

Let $[\tilde{u}]$ be the class of kinematically admissible displacement fields corresponding to $D+B$, B_1, and $u_i^*(x)$. Let $\Phi\{\tilde{u}\}$ be the functional of $\tilde{u}_i(x)$ defined by

$$\Phi\{\tilde{u}\} = \int_D W(\tilde{e})\,dV - \int_D f_i\,\tilde{u}_i\,dV - \int_{B_2} t_i^*\,\tilde{u}_i\,dA, \tag{6.41}$$

where $W(e)$ is the strain-energy density given by (6.19) and $\tilde{e}_{ij}(x)$ are the strains associated with $\tilde{u}_i(x)$ in the sense of (4.1).

Then

$$\Phi\{u\} = \min_{[\tilde{u}]} \Phi\{\tilde{u}\}, \tag{6.42}$$

and this absolute minimum is assumed by $\Phi\{\tilde{u}\}$ only if $\tilde{e}_{ij}(x) = e_{ij}(x)$ in $D+B$.

In outlining a proof of this theorem, we may confine our attention to the case of an infinite[20] D. Then $\Phi\{\tilde{u}\}$ is given by

$$\Phi\{\tilde{u}\} = \lim_{\varrho\to\infty} \Phi_\varrho\{\tilde{u}\}, \tag{6.43}$$

with

$$\Phi_\varrho\{\tilde{u}\} = \int_{R(\varrho)} W(\tilde{e})\,dV - \int_{R(\varrho)} f_i\,\tilde{u}_i\,dV - \int_{B_2} t_i^*\,\tilde{u}_i\,dA, \tag{6.44}$$

provided $R(\varrho)$ is once again the region whose boundary consists of B and the sphere $\Omega(\varrho)$ introduced in connection with the proof of Theorem 6.1. In view of (6.38), (6.40), and Theorem 5.1, the existence of $\Phi\{\tilde{u}\}$ is assured. By the present hypotheses and Theorem 5.1, $u_i(x)$ is a member of $[\tilde{u}]$.

Now let

$$u_i'(x) = \tilde{u}_i(x) - u_i(x), \qquad e_{ij}'(x) = \tilde{e}_{ij}(x) - e_{ij}(x). \tag{6.45}$$

From (6.44), (6.45), with the aid of (4.1), (4.2), (4.3), (6.37), (6.39), and the divergence theorem, follows the identity

$$\Phi_\varrho\{\tilde{u}\} - \Phi_\varrho\{u\} = \int_{R(\varrho)} W(e')\,dV + \int_{\Omega(\varrho)} \tau_{ij}\,n_j\,u_i'\,dA, \tag{6.46}$$

in which n_j designates the components of the outward unit normal of $\Omega(\varrho)$. But according to (6.45), (6.38), (6.40), and Theorem 5.1,

$$u_i'(x) = O(r^{-1}), \qquad e_{ij}'(x) = O(r^{-2}), \qquad \tau_{ij}(x) = O(r^{-2}). \tag{6.47}$$

Hence, passing to the limit as $\varrho\to\infty$ in (6.46), we have

$$\Phi\{\tilde{u}\} - \Phi\{u\} = \int_D W(e')\,dV. \tag{6.48}$$

Since $W(e)$ is positive definite, we draw from (6.48), (6.45) that

$$\Phi\{\tilde{u}\} \geqq \Phi\{u\} \tag{6.49}$$

and that the equality prevails in (6.49) if and only if $\tilde{e}_{ij}(x) = e_{ij}(x)$ in $D+B$, which is the desired result. A related minimum principle — also valid for both bounded and exterior regions, but applicable only to the first boundary-value problem (B_2 empty) — was established in [*6*]. We turn next to an extension of the principle of minimum complementary energy.

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[20] A proof valid for bounded regions may be found, for example, in HILL [*14*], p. 60 *et seq.*, except that in [*14*] the body forces are assumed to vanish.

Definition 6.2. *Let $\tilde{\tau}_{ij}(x)$ and $f_i(x)$ be a field of stress and a body-force field with the following properties:*

(a) *$\tilde{\tau}_{ij}(x)$ is of class $C^{(1)}$ in a regular region of space $D+B$;*

(b) *$\tilde{\tau}_{ij}(x)$ and $f_i(x)$ satisfy* (4.3) *in D;*

(c)
$$\tilde{\tau}_{ij} n_j = t_i^*(x) \quad \text{on } B_2, \tag{6.50}$$

where B_2 is a subset of B, n_j denotes the components of the outward unit normal of B, while $t_i^*(x)$ is a triplet of functions prescribed on B_2;

(d) *if D is infinite,*

$$\tilde{\tau}_{ij}(x) = O(r^{-2}), \tag{6.51}$$

$$\int_B \tilde{t}_i \, dA = L_i, \qquad \int_B \varepsilon_{ijp} \, x_j \tilde{t}_p \, dA = M_i \tag{6.52}$$

where $\tilde{t}_i = \tilde{\tau}_{ij} n_j$, while L_i and M_i are prescribed constants.

The totality of all such $\tilde{\tau}_{ij}(x)$ constitutes the class of statically admissible stress fields $[\tilde{\tau}]$ corresponding to $D+B$, $f_i(x)$, B_2, $t_i^*(x)$, and (if D is infinite) to L_i and M_i.

Theorem 6.6. *Let $D+B$, $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$, $f_i(x)$, $u_i^*(x)$, $t_i^*(x)$, as well as the elastic constants, satisfy the same hypotheses as in Theorem* 6.5, *except replace* (6.40) *with*

$$\tau_{ij}(x) = o(1), \qquad f_i(x) = o(r^{-2}), \tag{6.53}$$

$$\int_B t_i \, dA = L_i, \qquad \int_B \varepsilon_{ijp} \, x_j t_p \, dA = M_i, \tag{6.54}$$

where $t_i = \tau_{ij} n_j$, while L_i and M_i are prescribed constants.

Let $[\tilde{\tau}]$ be the class of statically admissible stress fields corresponding to $D+B$, $f_i(x)$, B_2, t_i^*, and (if D is infinite) to L_i and M_i. Let $\Psi\{\tilde{\tau}\}$ be the functional of $\tilde{\tau}_{ij}(x)$ defined by

$$\Psi\{\tilde{\tau}\} = \int_D W(\tilde{e}) \, dV - \int_{B_1} \tilde{t}_i u_i^* \, dA, \tag{6.55}$$

where $W(e)$ is the strain-energy density given by (6.19) *and $\tilde{e}_{ij}(x)$ are the strains associated with $\tilde{\tau}_{ij}(x)$ in the sense of* (4.5).

Then
$$\Psi\{\tau\} = \min_{[\tilde{\tau}]} \Psi\{\tilde{\tau}\}, \tag{6.56}$$

and this absolute minimum is assumed by $\Psi\{\tilde{\tau}\}$ only if $\tilde{\tau}_{ij}(x) = \tau_{ij}(x)$ in $D+B$.

It will again be sufficient to sketch a proof on the assumption that D is infinite[21]. In this instance $\Psi\{\tilde{\tau}\}$ is given by

$$\Psi\{\tilde{\tau}\} = \lim_{\varrho \to \infty} \Psi_\varrho\{\tilde{\tau}\} \tag{6.57}$$

with
$$\Psi_\varrho\{\tilde{\tau}\} = \int_{R(\varrho)} W(\tilde{e}) \, dV - \int_{B_1} \tilde{t}_i u_i^* \, dA, \tag{6.58}$$

provided $R(\varrho)$ has the same meaning as in the proof of Theorem 6.1. It is clear from (6.51), (6.19), and (4.5) that $\Psi\{\tilde{\tau}\}$ exists. Also, by hypothesis and Theorem 5.2,

$$u_i(x) = \hat{u}_i(x) + \mathring{u}_i(x), \tag{6.59}$$

[21] See Footnote No. 19.

where $\mathring{u}_i(x)$ is a rigid displacement field, and

$$\hat{u}_i(x) = O(r^{-1}), \quad e_{ij}(x) = O(r^{-2}), \quad \tau_{ij}(x) = O(r^{-2}). \tag{6.60}$$

In view of the present hypotheses and the last of (6.60), it is apparent that $\tau_{ij}(x)$ belongs to $[\tilde{\tau}]$.

Setting

$$\tau'_{ij}(x) = \tilde{\tau}_{ij}(x) - \tau_{ij}(x), \quad e'_{ij}(x) = \tilde{e}_{ij}(x) - e_{ij}(x), \tag{6.61}$$

we draw from (6.58), (6.19), and (4.5) that

$$\Psi_\varrho\{\tilde{\tau}\} - \Psi_\varrho\{\tau\} = \int_{R(\varrho)} W(e')\, dV + \int_{R(\varrho)} \tau'_{ij} e_{ij}\, dV - \int_{B_1} t'_i u^*_i\, dA. \tag{6.62}$$

Now, from the divergence theorem, (6.59), (4.1), and the fact that $\tau'_{ij}(x)$ evidently meets (4.3) with $f_i(x) = 0$,

$$\int_{R(\varrho)} \tau'_{ij} e_{ij}\, dV = \int_B t'_i \hat{u}_i\, dA + \int_{\Omega(\varrho)} t'_i \hat{u}_i\, dA, \tag{6.63}$$

provided $B + \Omega(\varrho)$ is the boundary of $R(\varrho)$. Further, because of (5.6), (6.52), (6.54), and (6.61),

$$\int_B t'_i \mathring{u}_i\, dA = 0, \tag{6.64}$$

so that (6.63), in view of (6.50), (6.39), the first of (6.61), and (6.59), implies

$$\int_{R(\varrho)} \tau'_{ij} e_{ij}\, dV = \int_{B_1} t'_i u^*_i\, dA + \int_{\Omega(\varrho)} t'_i \hat{u}_i\, dA. \tag{6.65}$$

By (6.62), and (6.65),

$$\Psi_\varrho\{\tilde{\tau}\} - \Psi_\varrho\{\tau\} = \int_{R(\varrho)} W(e')\, dV + \int_{\Omega(\varrho)} t'_i \hat{u}_i\, dA \tag{6.66}$$

or, on passing to the limit as $\varrho \to \infty$, and using (6.61), (6.60), (6.51),

$$\Psi\{\tilde{\tau}\} - \Psi\{\tau\} = \int_D W(e')\, dV. \tag{6.67}$$

The desired conclusion is immediate from (6.67) by virtue of (4.5), the second of (6.61), and the positive definiteness of $W(e)$.

Both Theorem 6.5 and Theorem 6.6 are easily generalized to accommodate also mixed-mixed boundary conditions. It is also worth mentioning that the uniqueness Theorem 6.3 is a corollary of the foregoing two minimum principles.

The results which have been established in this paper yield at once analogous extensions to exterior domains of the integral-representation theorems of classical elastostatics [22] and of Saint-Venant's principle in the formulation given in [*16*]. On the other hand, the generalization of the present results to anisotropic media or to infinite regions whose boundary extends to infinity, does not appear to be elementary.

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[22] See LOVE [*1*], art. 169. More explicit statements and proofs of these theorems can be found in [*15*].

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On the Scattering of Water Waves by a Circular Disk

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I. Introduction

The boundary-value problem describing the diffraction of two-dimensional water waves by a dock of finite width has not as yet been solved in explicit fashion. It has, however, received considerable attention. The general problem of diffraction by surface obstacles was investigated thoroughly by John [*1*] and later by Peters & Stoker [*2*]. An existence proof for the finite dock problem, using the calculus of variations, was given by Rubin [*3*]. Finally it was shown by the author [*4*] that the integral equation approach of [*1*] can be greatly simplified for the dock problem.

It is the purpose of this paper to point out that the results of [*3*] and [*4*] can be extended to the three-dimensional problem of diffraction by a circular dock. Indeed it proves possible to reduce the three-dimensional problem to essentially the same (two-dimensional) problems studied in [*3*] and [*4*]. The technique is an extension of that used by Heins and the author [*5*] to study the diffraction of sound waves by a circular disk. Briefly the solution is expanded in a Fourier series in the angular variable, and then the two-dimensional boundary problems for the Fourier coefficients are investigated. These coefficients satisfy rather complicated equations. However it turns out they can be expressed very simply in terms of certain harmonic functions of two variables, the latter being closely related to the values of the coefficients on the axis of the dock. When the boundary conditions on the coefficients are translated into conditions for the harmonic functions, one obtains essentially the problems of [*3*] and [*4*].

In addition to the fact that our methods yield an existence theorem together with a reasonably practical method for numerical solution of the disk problem, these methods seem to have an additional interest. It is the feeling of the author that many boundary-value problems for elliptic equations including as here those involving circular regions in three dimensions differ only in technical details from some analogous problems for harmonic functions of two variables. The connection is established by some integral representation of solutions of the equations in question. We use such a representation in Section V. Since a great deal is known about harmonic functions of two variables, one can hope to exploit this connection to obtain information about other equations. This paper is intended as an example of how one establishes the connection, and we illustrate its exploitation in the last section by finding the nature of the singularity produced at the edge of the dock.

II. General Formulation

An inviscid, incompressible fluid occupies the region $y<0$, $y=0$ being a free surface. We consider two problems. First suppose the motion is initially that of a plane progressive wave moving in the positive x-direction. A rigid circular disk is placed in the free surface and held in position, thus producing in addition to the original wave a diffracted wave pattern. All motions are assumed time-periodic, irrotational and of small amplitude. Suppose in the second case the fluid is initially at rest, and then motion is produced by moving the circular disk vertically with forced periodic vibrations. By combining the two problems one could determine the oscillations of a freely floating disk in an incoming wave.

If we divide all length dimensions by the radius a of the disk, the mathematical model of the above problems is as follows (see [1] and [2] for details):

Problem I (P. I). Find a function $u(r,\vartheta,y)$ continuous in $y\leqq 0$ such that

(A) $u_{rr}+r^{-1}u_r+r^{-2}u_{\vartheta\vartheta}+u_{yy}=0$ in $y<0$,

(B) $u(r,\vartheta,y)\sim A(\vartheta)\,r^{-\frac{1}{2}}\exp[ky+ikr]$ as $r\to\infty$, y bounded,

(C) $u_y(r,\vartheta,0)=g(r,\vartheta)$ for $0\leqq r<1$,

(D) $u_y(r,\vartheta,0)-k\,u(r,\vartheta,0)=0$ for $r<1$.

Here x and z are co-ordinates in the free surface and $r^2=x^2+z^2$, $\vartheta=\arctan z/x$. The parameter k is $\sigma^2 a/\gamma$ where σ is the frequency of the motion and γ the acceleration of gravity. g is to be an analytic function of x and z on $x^2+z^2\leqq 1$. For the diffraction problem we have $g=\exp(ir\cos\vartheta)$, while for the forced motion problem $g=1$.

We shall impose on the solution a regularity condition at the edge of the disk*. If c_ε denotes the half-tube, $(r-1)^2+y^2=\varepsilon$, $y<0$, we require

$$\lim_{\varepsilon\to 0}\iint_{c_\varepsilon} u\,u_n\,ds=0,\qquad n=\text{normal to } c_\varepsilon. \tag{II.1}$$

With this condition it is shown in [2] that the solution of (P. I) is unique.

We present now an integral formulation of (P. I). In a standard way we introduce a Green's function, $G(r,\varrho,\vartheta,\psi,y)$, by

$$\begin{aligned} &G(r,\varrho,\vartheta,\psi,y)\\ &=R^{-1}-k\int_y^\infty(\eta^2+\mu^2)^{-\frac{1}{2}}\exp[k(y-\eta)]\,d\eta+i\pi k\,e^{ky}H_0^{(1)}(k\mu)=R^{-1}+G_1,\\ &R^2=r^2+\varrho^2-2r\varrho\cos(\vartheta-\psi)+y^2,\qquad \mu^2=r^2+\varrho^2-2r\varrho\cos(\vartheta-\psi),\end{aligned} \tag{II.2}$$

$H_0^{(1)}(k\mu)$ denoting the Hankel function. It is easily verified that G is harmonic for $(r,\vartheta,y)\not\equiv(\varrho,\psi,0)$. Further, G satisfies (B) for ϱ bounded and

$$G_y-kG=\frac{\partial}{\partial y}\left(\frac{1}{R}\right). \tag{II.3}$$

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* The author believes this condition can be shown to follow from conditions (C) and (D) and the continuity of u in $y\leqq 0$.

For any function f satisfying a Hölder condition on $0 \leqq \varrho \leqq 1$ we introduce the operators

$$W(r,\vartheta,y;f) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 R^{-1} f(\varrho,\psi)\,\varrho\,d\varrho\,d\psi,$$

$$L(r,\vartheta,y;f) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(\varrho,\psi)\,G_1(r,\vartheta,y,\varrho,\psi)\,\varrho\,d\varrho\,d\psi,$$

$$U(r,\vartheta,y;f) = W + L.$$

Then for any such function f, U is a harmonic function in $y<0$, is continuous in $y \leqq 0$ and satisfies (B). Moreover from (II.3) we have

$$\text{(II.4)} \qquad \left(\frac{\partial}{\partial y} - k\right) U = \frac{\partial}{\partial y} W \quad \text{in} \quad y<0.$$

It is immediate that U satisfies (D). On the other hand, by a formula of potential theory, (II.4) yields

$$U_y(r,\vartheta,0;f) - k\,U(r,\vartheta,0;f) = f(r,\vartheta), \qquad 0 \leqq r < 1.$$

Accordingly $U(r,\vartheta,y;f)$ will be a solution of (P. I) if f is a solution ot the integral equation

$$\text{(II.5)} \qquad f(r,\vartheta) + k[W(r,\vartheta,0;f) + L(r,\vartheta,0;f)] = g(r,\vartheta) \qquad 0 \leqq r < 1.$$

It can be shown that $U(r,\vartheta,y;f)$ satisfies (II.1) for any f. If the homogeneous equation corresponding to (II.5) had a solution $f_0(r,\vartheta)$, then clearly $u^0(r,\vartheta,y) = U(r,\vartheta,y;f_0)$ would be a solution of (P. I) with the right side of (C) replaced by zero. Hence, by the uniqueness theorem quoted, $u_0(r,\vartheta,y) = 0$, and then by (II.4)

$$f_0(r,\vartheta) = \lim_{y\to 0}\left(\frac{\partial}{\partial y} - k\right) U(r,\vartheta;f_0) \equiv 0.$$

Thus the homogeneous equation can have no non-trivial solutions. From (II.2) we see that the kernel of equation (II.5) has a singularity of the form $\mu^{-1} + c\log\mu$; hence the Fredholm theory yields a (unique) solution $f(r,\vartheta)$ of equation (II.5). We summarize the results of this section.

Theorem 1. *The integral equation* (II.5) *has a unique Hölder-continuous solution f for any function g analytic on $0 \leqq r \leqq 1$, and the unique solution of problem* (P. I) *is given by*

$$u(r,\vartheta,y) = U(r,\vartheta,y;f).$$

III. Complementary Problems

In this section we use the methods of [4] to transform (P. I) into a simpler problem. We start with formula (II.4). Applying the operator $-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} + k\right)$ to both sides yields

$$\text{(III.1)} \qquad \left(-\frac{\partial^2}{\partial y^2} + k^2\right)\frac{\partial U}{\partial y} = -\frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial y} + k\right) W \quad \text{in} \quad y<0.$$

Now both U and W are harmonic functions in $y<0$; hence we may replace $\frac{\partial^2}{\partial y^2}$ by $-\Delta=-\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial}{\partial\vartheta^2}$, obtaining

$$\text{(III.2)} \qquad (\Delta+k^2)\frac{\partial U}{\partial y}=\Delta\left(\frac{\partial}{\partial y}+k\right)W \quad \text{in} \quad y<0.$$

Suppose now that $U(r,\vartheta,y;f)$ is a solution of (P. I). By (C) we see that the harmonic function

$$V(r,\vartheta,y)=\frac{\partial U}{\partial y}(r,\vartheta,y;f)$$

satisfies

$$V(r,\vartheta,0)=g(r,\vartheta), \qquad 0\leqq r<1.$$

Since g is an analytic function, it follows that V and hence U itself may be continued across the disk into $y>0$ as a harmonic function. From (III.2) then it follows that the harmonic function

$$\Delta\left(\frac{\partial}{\partial y}+k\right)W(r,\vartheta,y;f)$$

and hence W itself can be continued across the disk as a harmonic function. Thus U and W are analytic for $y\leqq 0$, $r<1$, and we may pass to the limit $y=0$ in (III.2), obtaining

$$\text{(III.3)} \qquad (\Delta+k^2)\,g(r,\vartheta)=\Delta(W_y+kW).$$

Now let $\Delta^{-1}(h)$ denote a *fixed* solution of the two-dimensional Poisson equation

$$\Delta w=h.$$

Then (III.3) yields

$$\text{(III.4)} \qquad \left(\frac{\partial}{\partial y}+k\right)W=[1+k^2\Delta^{-1}](g)+H, \qquad \Delta H=0.$$

The operator W satisfies

$$W_y(r,\vartheta,0;f)=0, \qquad r>1,$$

so that (III.4) suggests that we should consider the following complementary problem.

Problem II (P. II). Find a function $v(r,\vartheta,y)$ continuous in $y\leqq 0$ such that

(A′) $v_{rr}+r^{-1}v_r+r^{-2}v_{\vartheta\vartheta}+v_{yy}=0$ in $y<0$,

(B′) $v\to 0$ as $r^2+y^2\to\infty$,

(C′) $v_y(r,\vartheta,0)+k\,v(r,\vartheta,0)=h(r,\vartheta)$, $r<1$,

(D′) $v_y(r,\vartheta,0)=0$, $r>1$.

If we further require the condition (II.1) for the function v, one readily finds, using Green's theorem, that the solution of (P. II) is unique. Also we can prove the existence of a solution, as for (P. I), by finding a unique $f(r,\vartheta)$ satisfying

$$\text{(III.5)} \qquad f(r,\vartheta)+k\,W(r,\vartheta,0;f)=h(r,\vartheta), \qquad r<1$$

and setting

$$v(r,\vartheta,y)=W(r,\vartheta,y;f).$$

Note that

$$f(r, \vartheta) = v_y(r, \vartheta, 0). \tag{III.6}$$

The problem (P. II) is simpler than (P. I) because of the simplified kernel in the integral equation. It would be advantageous to be able to assert that the solution of (P. II) yields the solution of (P. I). However, so far we have shown only that the solution of (P. I) yields a solution of (P. II). Suppose now, as suggested by (III.4), we find a solution of (P. II) for $h(r, \vartheta) = [1 + k^2 \Delta^{-1}]\, g$ and set $f(r, \vartheta) = v_y(r, \vartheta, 0)$. Then by (III.2) we could deduce only that

$$(\Delta + k^2)\, U_y(r, \vartheta, 0; f) = \Delta[1 + k^2 \Delta^{-1}]\, g(r, \vartheta) = (\Delta + k^2)\, g(r, \vartheta)$$

from which (C) does not follow.

The difficulty here is of course that the null space of the operator $(\Delta + k^2)$ is infinite-dimensional. To circumvent this problem we try to find approximate solutions in spaces in which the null space of $\Delta + k^2$ has finite dimension, and this leads us to a Fourier series decomposition.

Theorem 2. *If the function* $g(r, \vartheta)$ *of* (P. I) *has the form,*

$$g(r, \vartheta) = G^N(r, \vartheta) = \sum_{n=0}^{N} a_n(r) \cos n\vartheta,$$

there exists a uniquely determined function $h(r, \vartheta)$ *such that if* v *is the solution of* (P. II), *then the solution of* (P. I) *is given by*

$$u(r, \vartheta, y) = U(r, \vartheta, y; f), \qquad f(r, \vartheta) = v_y(r, \vartheta, 0).$$

Let $w_n(r)$ be solutions of

$$w_n'' + r^{-1} w_n' - n^2 r^{-2} w_n = a_n(r) \tag{III.7}$$

with $w_n(0)$ bounded. We find

$$w_n(r) = r^n \int_0^r \varrho^{-2n-1}\, d\varrho \int_0^{\varrho} a_n(\tau)\, \tau^{n+1}\, d\tau.$$

Convergence of these integrals is assured by a lemma of [5] which shows that the analyticity of $g(r, \vartheta)$ implies that the $a_n(\tau)$ must be $O(\tau^n)$ near $\tau = 0$. Let $h^N(r, \vartheta)$ denote the sums

$$h^N(r, \vartheta) = \sum_{n=0}^{N} \big(a_n(r) + k^2 w_n(r)\big) \cos n\vartheta.$$

Let $\bar{v}(r, \vartheta, y)$ be the solution of (P. II) for $h(r, \vartheta) = h^N(r, \vartheta)$ and $v^n(r, \vartheta, y)$, $n = 0, 1, \ldots, N$ be solutions of (P. II) for $h(r, \vartheta) = r^n \cos n\vartheta$, respectively. Following our previous remarks, we have

$$\bar{v} = W(r, \vartheta, y; \bar{f}), \qquad v^n = W(r, \vartheta, y; f^n),$$
$$\bar{f} + W(r, \vartheta, 0; \bar{f}) = h^N(r, \vartheta),$$
$$f^n + W(r, \vartheta, 0; f^n) = r^n \cos n\vartheta.$$

By straightforward calculation we may verify that the functions $\bar{f}$ and f^n must have the form

$$\bar{f} = \sum_{n=0}^{N} G_n(r) \cos n\vartheta, \qquad f^n = F_n(r) \cos n\vartheta.$$

We show that the (unique) solution of (P. I) with $g(r,\vartheta)$ equal to $G^N(r,\vartheta)$ can be written in the form

$$u(r,\vartheta,y) = U(r,\vartheta,y;\bar{f}) + \sum_{n=0}^{N} A_n U(r,\vartheta,y;f^n)$$

for some appropriate constants A_n. From (III.2) we have for such a u,

$$(\Delta + k^2)\, u_y(r,\vartheta,0) = \Delta\left(\frac{\partial}{\partial y} + k\right)\left\{W(r,\vartheta,0;\bar{f}) + \sum_{n=0}^{N} A_n W(r,\vartheta,0;f^n)\right\}$$
$$= \Delta\left\{h^N(r,\vartheta) + \sum_{n=0}^{N} A_n r^n \cos n\vartheta\right\}.$$

But $\Delta(r^n \cos n\vartheta) = 0$, and by (III.7)

$$\Delta\left(h^N(r,\vartheta)\right) = \Delta \sum_{n=0}^{N} \left(a_n(r) + k^2 w_n(r)\right) \cos n\vartheta = (\Delta + k^2) \sum_{n=0}^{N} a_n(r) \cos n\vartheta .$$

Hence

$$(\Delta + k^2)\, u_y(r,\vartheta,0) = (\Delta + k^2)\, G^N(r,\vartheta),$$

and accordingly

$$u_y(r,\vartheta,0) = G^N(r,\vartheta) + K(r,\vartheta), \tag{III.8}$$

where $K(r,\vartheta)$ is a solution of

$$(\Delta + k^2)\, K = 0 . \tag{III.9}$$

Again it is not difficult to verify that

$$U\left(r,\vartheta,0;h(\varrho)\cos n\psi\right) = B(r)\cos n\vartheta .$$

It follows that the function $U(r,\vartheta,0;\bar{f}) + \sum_{n=0}^{N} A_n U(r,\vartheta,0;f^n)$ has only N non-zero Fourier coefficients. Since this fact is also true of $G^N(r,\vartheta)$, we see by (III.8) that it must be true of the function $K(r,\vartheta)$. But $K(r,\vartheta)$ is also a solution of (III.9), and hence we must have

$$K(r,\vartheta) = \sum_{n=0}^{N} \alpha_n J_n(kr) \cos n\vartheta ,$$

the J_n's being regular Bessel functions.

The course is now clear. We choose the constants A_n so as to make the coefficients α_n in $K(r,\vartheta)$ all zero. Since $J_n^{(n)}(0) = \frac{1}{2^n}$ and $J_k^{(n)}(0) = 0$ for $n < k$, the conditions

$$\frac{\partial^n}{\partial r^n} K = 0 \quad \text{at} \quad r = 0, \qquad n = 0, 1, \ldots, N,$$

will insure that $K(r,\vartheta) = 0$. Then according to (III.8) we choose the constants A_n as solutions of the equations

$$\frac{\partial^n}{\partial r^n} U(0,0,0;\bar{f}) + \sum_{n=0}^{N} A_n \frac{\partial^n}{\partial r^n} U(0,0,0;f_n) = \frac{\partial G^n}{\partial r^n}(0,0), \quad n = 0, 1, \ldots, N. \tag{III.10}$$

This is a non-homogeneous system of linear equations for the constants A_n. It will always have a solution unless the associated homogeneous system should

have a non-trivial solution. Suppose the latter were the case, and let A_n^0 denote that solution. Then the above argument shows that

$$u^0(r, \vartheta, y) = \sum_{n=0}^{N} A_n^0 U(r, \vartheta, y; f^n)$$

is a solution of (P. I) with the right side of (C) replaced by zero. Hence by the uniqueness theorem $u^0(r, \vartheta, y) = 0$. Recalling that

$$U_y(r, \vartheta, 0; f) - k\,U(r, \vartheta, 0; f) = f(r, \vartheta), \qquad 0 \leqq r < 1,$$

we infer then that

$$0 \equiv u_y^0(r, \vartheta, 0) - k\,u^0(r, \vartheta, 0) = \sum A_n^0 f^n.$$

But by construction we have

$$\sum_{n=0}^{N} A_n^0 f^n + W\Big(r, \vartheta, 0; \sum_{n=0}^{N} A_n^0 f^n\Big) = \sum_{n=0}^{N} A_n^0 r^n \cos n\vartheta.$$

Combining the last two formulas, we conclude

$$\sum_{n=0}^{N} A_n^0 r^n \cos n\vartheta \equiv 0$$

from which $A_n^0 = 0$ $n = 0, 1, \ldots, N$. The contradiction shows that the system (III.10) always has a solution. The proof of Theorem 2 is now complete.

We make some further remarks concerning the coefficients in (III.10) in the Appendix, but here we have at least in principle reduced the solution of (P. I) with right side of (C) equal to $G^N(r, \vartheta)$ to the solution of (P. II). Translated into integral equations this means that we can express the solution of (II.5), with right side equal to $G^N(r, \vartheta)$, in terms of solutions of (III.5).

Let $f^N(r, \vartheta)$ be the solution of (II.5) with right side equal to $G^N(r, \vartheta)$. Since the functions $G^N(r, \vartheta)$ converge uniformly to $g(r, \vartheta)$, one may expect that the functions $f^N(r, \vartheta)$ converge to a function $f(r, \vartheta)$ which is a solution of (II.5) itself. We verify now that such is indeed the case.

Theorem 3. *Let $G^N(r, \vartheta)$ be a sequence of functions converging uniformly to $g(r, \vartheta)$ in $0 \leqq r \leqq 1$, and suppose the functions $f^N(r, \vartheta)$ are solutions of the integral equations*

$$f^N(r, \vartheta) + k[W(r, \vartheta, 0; f^N) + L(r, \vartheta, 0; f^N)] = G^N(r, \vartheta), \qquad 0 \leqq r < 1.$$

Then the functions $f^N(r, \vartheta)$ converge to a solution $f(r, \vartheta)$ of the equation

$$f(r, \vartheta) + k[W(r, \vartheta, 0; f) + L(r, \vartheta, 0; f)] = g(r, \vartheta), \qquad 0 \leqq r < 1.$$

We need first a lemma expressing the complete continuity of the operator U.

Lemma. *Let $f^n(r, \vartheta)$ be a uniformly bounded sequence of functions continuous and satisfying a Hölder condition on $0 \leqq r \leqq 1$. Then the functions $U(r, \vartheta, 0; f^n)$ form an equi-continuous family.*

For ease of writing we denote the integration points $(\varrho, \psi, 0)$ by Q, the variable points $(r, \vartheta, 0)$ by P, and the distance from P to Q by PQ. Now by (II.2)

$$G = \frac{1}{PQ} + K(P, Q)$$

where K is square integrable. Since L was defined on p. 122 as an integral operator with square integrable kernel, it follows easily by Schwarz' inequality and the boundedness of f^n that the difference

$$L(P_1; f^n) - L(P_2; f^n)$$

can be made uniformly small by choosing $P_1 P_2$ sufficiently small. The difficulties arise from the term $\frac{1}{PQ}$ in G. If $|f^n| \leqq M$, we have

$$\text{(III.11)}\qquad \begin{aligned} 2\pi\,|W(P_1; f^n) - W(P_2; f^n)| &= \left|\iint\limits_{r<1} f(Q)\left(\frac{1}{P_1Q} - \frac{1}{P_2Q}\right) d s_Q\right| \\ &\leqq M \iint\limits_{r<1} \left|\frac{1}{P_1Q} - \frac{1}{P_2Q}\right| d s_Q. \end{aligned}$$

Now let $\delta = P_1 P_2$, and let s_1 denote the intersection of a disk of radius 2δ and center P_1 with the disk $r<1$. We have

$$\iint\limits_{r<1} \left|\frac{1}{P_1Q} - \frac{1}{P_2Q}\right| d s_Q = \iint\limits_{s_1} + \iint\limits_{(r<1)-s_1}.$$

By a known result of potential theory ([7], p. 129), for any region s of the plane, of area A, we have

$$\iint\limits_{s} \frac{d s_Q}{PQ} \leqq 2(\pi A)^{\frac{1}{2}}$$

independently of $P \in s$. Thus we find

$$\text{(III.12)}\qquad \iint\limits_{s_1} \left|\frac{1}{P_1Q} - \frac{1}{P_2Q}\right| d s_Q \leqq 4(4\pi^2\delta^2)^{\frac{1}{2}} = 8\pi\delta.$$

Suppose on the other hand that $Q \in [(r<1) - s_1]$. Then we have

$$|P_2Q - P_1Q| \leqq \delta, \qquad P_2Q \geqq P_1Q - \delta,$$

and

$$\iint\limits_{(r<1)-s_1} \left|\frac{1}{P_1Q} - \frac{1}{P_2Q}\right| d s_Q = \iint \left|\frac{P_2Q - P_1Q}{P_1Q\,P_2Q}\right| d s_Q \leqq \delta \iint \frac{d s_Q}{P_1Q(P_1Q - \delta)}.$$

Let us choose new polar co-ordinates (ϱ, α) with origin at P_1, $P_1Q = \varrho$. Then our last inequality becomes

$$\text{(III.13)}\qquad \iint\limits_{(r<1)-s_1} \left|\frac{1}{P_1Q} - \frac{1}{P_2Q}\right| d s_Q \leqq \delta \int_0^{2\pi}\int_{2\delta}^{R(\alpha)} \frac{d\varrho\, d\alpha}{\varrho - \delta} \leqq \delta \int_0^{2\pi}\int_{2\delta}^{1} \frac{d\varrho\, d\alpha}{\varrho - \delta} = 2\pi\delta\left(\log\frac{1-2\delta}{2\delta}\right).$$

If we enter the estimates (III.12) and (III.13) in the inequality (III.11), we obtain

$$2\pi\,|W(P_1; f^n) - W(P_2; f^n)| \leqq M\delta\left(8\pi + 2\pi\log\frac{1-2\delta}{2\delta}\right).$$

The right side becomes uniformly small with δ, so that the family $W(P; f^n)$ is equi-continuous, and the lemma is proved.

We return now to the equation

$$f^n(P) + U(P; f^n) = G^n(P), \tag{III.14}$$

in obvious notation. We wish to prove that the sequence f^n is uniformly bounded and equi-continuous. Suppose first that the f^n are uniformly bounded. Given any $\varepsilon > 0$ we can find by the lemma a δ_1 so small that $PP_1 < \delta_1$ implies

$$|U(P; f^n) - U(P_1; f^n)| < \tfrac{1}{2}\varepsilon$$

uniformly as to n. From the uniform convergence of the $G^N(P)$ to $g(r, \vartheta)$ it is clear that we can also find a δ_2 such that for $PP_1 < \delta_2$

$$|G(P) - G(P_1)| < \tfrac{1}{2}\varepsilon.$$

Then if $\delta = \min(\delta_1, \delta_2)$, we have

$$|f^n(P) - f^n(P_1)| < \varepsilon \quad \text{for} \quad PP_1 < \delta \quad \text{uniformly in } n.$$

It follows that the f_n are equi-continuous; hence a subsequence converges uniformly to $f(r, \vartheta)$, a continuous function. It is well known (and proved essentially as the lemma) that an integral operator of the form of U is continuous*. Hence we have

$$g(r, \vartheta) = \lim_{N\to\infty} G^N(r, \vartheta) = \lim_{N\to\infty} \left(f^N + U(r, \vartheta, 0; f^N)\right) = f + U(r, \vartheta, 0; f).$$

We must still consider the possibility that the f^n are not uniformly bounded. Let σ^n be the maximum of $f^n(r, \vartheta)$ on $r < 1$ and $g^n(r, \vartheta) = f^n/\sigma^n$. Then

$$g^n(P) + U(P, g^n) = G^n/\sigma^n. \tag{III.15}$$

If the σ^n are not bounded, there exists a subsequence n_i tending to infinity so that G^{n_i}/σ^{n_i} tends to zero. We could then repeat the steps of the preceding case to infer that there is a further subsequence $\{g^{n_k}\}$ converging to a continuous function g which by (III.15) must clearly be identically zero; that is,

$$\lim_{k\to\infty} g^{n_k}(r, \vartheta) = 0.$$

But $\max g^{n_k}(r, \vartheta) = 1$ for all n; hence we have a contradiction, showing the f^n must be uniformly bounded. Theorem 3 is thus proved.

The results of this section show that at least up to a convergent approximation method we can reduce the solution of (P. I) to that of (P. II). In Section IV we point out that the latter can be formulated as a variational problem for a positive definite functional. In Section V we present an alternative method of attack for (P. II).

IV. A Variational Problem

If one attempts to formulate a variational problem equivalent to (P. I), two difficulties are encountered. First the functional involved is not positive, and second the conditions at infinity are such that this functional is not bounded. These facts are to be expected since the problem, like that of diffraction of acoustic waves, involves infinite energy. The same difficulties were observed by Rubin [3] in two dimensions. He was able to overcome them by considering

* Continuous with norm taken as the maximum of the function on $r \leqq 1$.

a complementary problem related to our (P. II). We show in this section that the same technique applies in the three-dimensional case.

Let F denote the class of functions $v(r, \vartheta, y)$ which satisfy the following conditions:

(1) v continuous in $y \leqq 0$,

(2) v of class $C^{(2)}$ in $y \leqq 0$ for $(r, y) \not\equiv (1, 0)$,

(3) $v \to 0$ as $r^2 + y^2 \to \infty$,

(4) v satisfies condition (II.1),

(5) $F[v] = k \iiint\limits_{y<0} (\operatorname{grad} v)^2 \, d\tau + \iint\limits_{\substack{r<1 \\ y=0}} (k v - h)^2 \, d\sigma < \infty$.

The volume integral in (5) is the limit of an integral over a finite volume. Letting C_ε denote the half-tube $(r-1)^2 + y^2 = \varepsilon$, $y \leqq 0$ of condition (II.1), Γ_R the hemisphere $r^2 + y^2 = R^2$, $y \leqq 0$, and $D_{\varepsilon,R}$ the region bounded by C_ε, Γ_R and the plane $y = 0$, we can define the volume integral by

$$\iiint\limits_{y<0} (\operatorname{grad} v)^2 \, d\tau = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \iiint\limits_{D_{\varepsilon,R}} (\operatorname{grad} v)^2 \, d\tau .$$

We now pose a variational problem.

Variational problem. Among all $v \in F$ find that one minimizing $F[v]$.

Theorem 4. *The variational problem and* (P. II) *possess the same uniquely determined solution* v.

Suppose that v is a solution of the variational problem. Let η be any function of F which vanishes identically outside the region $D_{\varepsilon,R}$ above. Then the vanishing of the first variation of F at v implies, by Green's theorem,

$$-\iiint\limits_{D_{\varepsilon,R}} (\eta \nabla^2 v) \, d\tau + \int\limits_0^{1-\varepsilon} \int\limits_0^{2\pi} (v_y + k v - h) \, \eta \, r \, d\vartheta \, dr + \int\limits_{1+\varepsilon}^{R} \int\limits_0^{2\pi} v_y \eta \, r \, d\vartheta \, dr = 0 .$$

Since η, ε and R are arbitrary, it follows that v satisfies conditions (A′), (C′) and (D′) of (P. II). The continuity of v in $y \leqq 0$ and conditions (B′) and (II.1) follow from the fact that $v \in F$. This shows that every solution of the variational problem is also a solution of (P. II). Since we know, however, that the solution of (P. II) is unique, it follows that there can be at most one solution of the variational problem.

We have shown in Section III that (P. II) does possess a solution v which satisfies conditions (1)—(4) of the variational problem. Since it was found in Section III that the solution v could be written in the form

$$v(r, \vartheta, y) = W(r, \vartheta, y; f)$$

for some suitable function f, condition (3) may be sharpened to

(IV.1) $\quad v = O\left((r^2 + y^2)^{-\frac{1}{2}}\right), \quad \operatorname{grad} v = O\left((r^2 + y^2)^{-\frac{3}{2}}\right) \quad$ as $\quad r^2 + y^2 \to \infty$.

Accordingly we can show that $F[v]$ for this (unique) solution of (P. II) is finite. From Green's theorem and the fact that v is harmonic in $y < 0$ we have

$$\iiint\limits_{D_{\varepsilon,R}} (\operatorname{grad} v)^2 \, d\tau = \iint\limits_{\partial D_{\varepsilon,R}} v \frac{\partial v}{\partial n} \, d\sigma ,$$

where $\partial D_{\varepsilon,R}$ denotes the boundary of $D_{\varepsilon,R}$. Using conditions (C′) and (D′) of (P. II) thus yields

$$\iiint\limits_{D_{\varepsilon,R}} (\operatorname{grad} v)^2\, d\tau = \iint\limits_{C_\varepsilon} v\, \frac{\partial v}{\partial n}\, d\sigma + \iint\limits_{\Gamma_R} v\, \frac{\partial v}{\partial n}\, d\sigma + \int\limits_0^{1-\varepsilon} \int\limits_0^{2\pi} v\,(h - k\,v)\, r\, dr\, d\vartheta\,.$$

Since v is continuous in $y \leqq 0$, the third integral has a limit as $\varepsilon \to 0$. The first and second integrals approach 0 as $\varepsilon \to 0$ and $R \to \infty$ by conditions (II.1) and (IV.1) respectively. Hence $F[v]$ is finite. Thus the solution v of (P. II) is an admissible function for the variational problem. We show that it actually yields the minimum for $F[v]$. Let η be any other admissible function; then (with the same limit interpretation of the volume integrals as before)

$$\text{(IV.2)}\qquad \begin{aligned} F[v+\eta] = F[v] &+ k^2 \iint\limits_{\substack{r<1\\ y=0}} \eta^2\, d\sigma + k \iiint\limits_{y<0} (\operatorname{grad}\eta)^2\, d\tau + \\ &+ 2k \iiint\limits_{y<0} (\operatorname{grad} v \cdot \operatorname{grad}\eta)\, d\tau + 2k \iint\limits_{\substack{r<1\\ y=0}} (k\,v - h)\, d\sigma\,. \end{aligned}$$

From Green's theorem and conditions (II.1) and (IV.1) we have

$$2k \iiint\limits_{y<0} (\operatorname{grad} v \cdot \operatorname{grad}\eta)\, d\tau = 2k \iint\limits_{\substack{r<1\\ y=0}} \eta\, v_y\, d\sigma\,.$$

Thus by (C′) of (P. II) equation (IV.2) yields

$$F[v+\eta] \geqq F[v]\,,$$

equality holding only if $\eta \equiv 0$. This shows that v is a solution of the variational problem and completes the proof of Theorem 4.

V. Solution of the Complementary Problem

If we study the solutions of (P. II) necessary to solve (P. I), we see that it suffices to consider (P. II) for

$$\text{(V.1)}\qquad h(r,\vartheta) = r^n\, h_n(r) \cos n\vartheta\,,$$

the functions $h_n(r)$ being analytic functions of r^2. We show in this section that this problem may be further reduced to a boundary-value problem for harmonic functions in two variables. We state the results here.

Problem III (P. III). Find a function $\varphi(y, \varrho)$ continuous in $y \leqq 0$ such that

(A″) $\varphi_{yy} + \varphi_{\varrho\varrho} = 0$ in $y < 0$,

(B″) $\varphi \to 0$ as $\varrho^2 + y^2 \to \infty$,

(C″) $\varphi_y(0, \varrho) + k\,\varphi(0, \varrho) = m(\varrho)$
$-1 < \varrho < +1$, $m(\varrho)$ analytic in $-1 \leqq \varrho \leqq +1$,

(D″) $\varphi(0, \varrho) = 0$ $|\varrho| > 1$.

Theorem 5. *Problem* (P. III) *possesses a unique solution*

$$\varphi(y, \varrho) = \Phi(y, \varrho; m)$$

for each $m(\varrho)$.

Theorem 6. *Let* $v(r, \vartheta, y)$ *be the solution of* (P. II) *for*

$$h(r,\vartheta) = r^n h_n(r) \cos n\vartheta, \qquad h_n(r) \text{ an analytic function of } r^2.$$

Then there exists a unique function $m(\varrho)$, *analytic in* $-1 \leq \varrho \leqq +1$, *such that*

$$v(r,\vartheta,y) = 2c_n r^{-n}\left[\int_0^r (r^2-\varrho^2)^{n-\frac{1}{2}} \Phi(y,\varrho;m)\,d\varrho\right]\cos n\vartheta, \tag{V.2}$$

the constants c_n *being independent of* $m(\varrho)$.

We prove Theorem 6 first. That a solution of (P. II) exists follows from Section III. Also as remarked in that section the solution, for the choice (V.1) of h, must have the form

$$v(r,\vartheta,y) = W(r,y)\cos n\vartheta.$$

Since, however, $v(r,\vartheta,y)$ is harmonic and hence analytic as a function of x, y, z, it follows from a lemma in [5] quoted above that

$$W(r,y) = r^n w(r,y).$$

From (A′) and (C′) follows

$$w_{rr} + (2n+1)\,r^{-1} w_r + w_{yy} = 0 \quad \text{in } y<0, \tag{V.3}$$

$$w_y(r,0) + k\,w(r,0) = h_n(r), \qquad 0<r<1. \tag{V.4}$$

Let us recall that $v(r,\vartheta,y)$ can be represented in the form

$$v(r,\vartheta,y) = W(r,\vartheta,y;f), \qquad f(r,\vartheta) = v_y(r,\vartheta,0).$$

Accordingly we have

$$f(r,\vartheta) = r^n F(r)\cos n\vartheta,$$

$$r^n w(r,y)\cos n\vartheta = \frac{1}{2\pi}\cos n\vartheta \int_0^{2\pi}\int_0^1 \frac{F(\varrho)\,\varrho^{n+1}\cos n\psi\,d\varrho\,d\psi}{[r^2+\varrho^2-2r\varrho\cos+y^2]^{\frac{1}{2}}}.$$

From this formula follows the relation

$$n!\,w(0,y) = \frac{\partial}{\partial(y^2)^n}\int_0^1 \frac{F(\varrho)\,\varrho^{n+1}\,d\varrho}{[\varrho^2+y^2]^{\frac{1}{2}}}, \qquad y<0. \tag{V.5}$$

A proof of a more general result including (V.5) is given in [5].

So far all quantities introduced have been real. However $w(r,y)$, being $r^{-n}v$, is analytic for real $y<0$. The formula (V.5) serves a definition of $w(0,y)$ for complex $y=\xi+i\eta$, $\xi<0$. Furthermore we can use (V.5) to continue $w(0,y)$ as a function of complex y. For $y=i\eta$, $|\eta|>1$ we have $\frac{\partial}{\partial(y^2)} = -\frac{\partial}{\partial\eta^2}$, and from (V.5)

$$\int_0^1 \frac{F(\varrho)\,\varrho^{n+1}\,d\varrho}{[\varrho^2+y^2]^{\frac{1}{2}}} = -i\int_0^1 \frac{F(\varrho)\,\varrho^{n+1}}{[\eta^2-\varrho^2]}\,d\varrho.$$

Hence,

$$\operatorname{Re} w(0,i\eta) = 0 \quad \text{on} \quad |\eta|>1. \tag{V.6}$$

It follows that $w(0,y)$ may be continued across $\xi=0$, $|\eta|>1$ into $\xi>0$ so as to become an analytic function on the complex y-plane slit from $-i$ to $+i$

along the imaginary axis. Since $w(0, y)$ is real for y real, we have also

$$\overline{w(0, y)} = w(0, \bar{y}). \tag{V.7}$$

We write

$$w(0, \xi + i\eta) = \varphi(\xi, \eta) + i\psi(\xi, \eta).$$

$\varphi(\xi, \eta)$ is then harmonic in $\xi < 0$, and from (V.5) and (V.6) we have

$$\varphi(0, \eta) = 0 \quad \text{on} \quad |\eta| > 1, \tag{V.8}$$

$$\varphi(\xi, -\eta) = \varphi(\xi, \eta), \qquad \psi(\xi, -\eta) = -\psi(\xi, \eta). \tag{V.9}$$

Finally from (V.5) we observe that

$$\varphi(\xi, \eta) \to 0 \quad \text{as} \quad \xi^2 + \eta^2 \to \infty. \tag{V.10}$$

We have shown now that $\varphi(\xi, \eta)$ satisfies conditions (A″), (B″) and (D″) of (P. III). In order to obtain condition (C″) and to work toward the specific formula (V.2) we make use of the following fact concerning equation (V.3). It is well known (see for example [7]) that solutions of (V.3) are uniquely determined wherever they exist by their values on any portion of the axis of symmetry, $r = 0$, provided those values be analytic. The answer can be written down explicitly in the "Laplace" integral

$$w(r, y) = c_n r^{-2n} \int_{-r}^{+r} (r^2 - \varrho^2)^{n-\frac{1}{2}} w(0, y + i\varrho)\, d\varrho, \qquad c_n = \Gamma(n)/\Gamma(n + \tfrac{1}{2})\, \Gamma(\tfrac{1}{2}).$$

This formula is valid in our case for $y < 0$ since we have observed that $w(0, \xi + i\eta)$ is analytic for $\xi < 0$; by (V.9), it may be rewritten as

$$w(r, y) = c_n r^{-2n} \int_0^r (r^2 - \varrho^2)^{n-\frac{1}{2}} \varphi(y, \varrho)\, d\varrho. \tag{V.11}$$

Substituting in condition (V.4), we find

$$r^{2n} h_n(r) = 2c_n \int_0^r (r^2 - \varrho^2)^{n-\frac{1}{2}} [\varphi_y(0, \varrho) + k\varphi(0, \varrho)]\, d\varrho, \qquad 0 \leqq r < 1.$$

Setting

$$r^2 = \alpha, \qquad \varrho^2 = \beta, \qquad [\varphi_y(0, \beta^{\frac{1}{2}}) + k\varphi(0, \beta^{\frac{1}{2}})] = x(\beta),$$

we express the last equation in the form

$$\alpha^n h_n(\alpha^{\frac{1}{2}}) = 2c_n \int_0^\alpha (\alpha - \beta)^{n-\frac{1}{2}} x(\beta)\, d\beta, \qquad 0 \leqq \alpha < 1,$$

or, differentiating n times with respect to α,

$$\Lambda(\alpha) \equiv \frac{d^n}{d\alpha^n} (\alpha^n h_n(\alpha^{\frac{1}{2}})) = \frac{n!}{\pi} \int_0^\alpha (\alpha - \beta)^{-\frac{1}{2}} x(\beta)\, d\beta, \qquad 0 \leqq \alpha < 1. \tag{V.12}$$

Equation (V.12) is an Abel equation which can be inverted to yield

$$x(\alpha) = \frac{1}{n!} \frac{d}{d\alpha} \int_0^\alpha (\alpha - \beta)^{-\frac{1}{2}} \Lambda(\beta)\, d\beta, \qquad 0 \leqq \alpha < 1,$$

or, setting $D = \frac{1}{\tau}\frac{d}{d\tau}$ and returning to the original variables,

$$\varphi_y(0,\varrho) + k\,\varphi(0,\varrho) = m(\varrho) = \frac{1}{2^n n!}\frac{d}{d\varrho}\int_0^{\varrho} (\varrho^2 - \tau^2)^{-\frac{1}{2}}\,\tau\, D^n\big(\tau^{2n} h_n(\tau)\big)\, d\tau, \tag{V.13}$$

$$0 \leqq \varrho < 1.$$

Since by (V.9) $\varphi(y,\varrho)$ is an even function of ϱ, (V.13) must also hold on $-1 < \varrho \leqq 0$ *. It is proved in Section VI that $m(\varrho)$ is an analytic function of ϱ. The last member of (V.13) yields the desired function $m(\varrho)$ of Theorem 6, and equation (V.11) yields formula (V.2). Thus Theorem 6 has been proved.

Having shown that (P. II) can be reduced to (P. III), we turn to Theorem 5, which asserts the existence and uniqueness of the solution of (P. III). We omit the rather simple proof of uniqueness, which, like that of (P. II), follows directly from Green's theorem once account is taken of the singularities at the edges, $(y,\varrho) = (0, \pm 1)$. The necessary estimates near these edges are contained in formula (VI.5) of the next section.

In order to construct a solution we proceed as with problem (P. II). We seek a solution in the form

$$\varphi(y,\varrho) = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-1}^{+1} \mu(t)\log[(\varrho - t)^2 + y^2]\, dt \tag{V.14}$$

for some function $\mu(t)$, Hölder-continuous on $-1 \leqq t \leqq +1$. This form insures that conditions (A″), (B″) and (D″) are satisfied no matter what the choice of μ. By a familiar formula we have also

$$\varphi(0,\varrho) = \lim_{y\to 0}\left\{-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-1}^{+1}\mu(t)\log[(\varrho - t)^2 + y^2]\, dt\right\} = \mu(t), \qquad -1 < t < +1.$$

Assume now that $\mu'(t)$ exists and is Hölder-continuous on $-1 < t < +1$ and that

$$\mu(+1) = \mu(-1) = 0. \tag{V.15}$$

Integrating by parts, we have

$$\begin{aligned}\frac{\partial^2}{\partial y^2}\int_{-1}^{+1}\mu(t)\log[(\varrho - t)^2 + y^2]\, d &= -\int_{-1}^{+1}\mu(t)\frac{\partial^2}{\partial t^2}\log[(\varrho - t)^2 + y^2]\, dt\\ &= -\int_{-1}^{+1}\frac{\mu'(t)\,(\varrho - t)}{(\varrho - t)^2 + y^2}\, dt.\end{aligned}$$

Thus if the form (V.14) is to yield a solution of (P. III), $\mu(t)$ must be chosen so that

$$\mu(\varrho) + \frac{1}{\pi}\int_{-1}^{+1}\frac{\mu'(t)}{\varrho - t}\, dt = m(\varrho), \qquad -1 < \varrho < +1, \tag{V.16}$$

the integral to be interpreted as the principal value.

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* Since $h_n(\tau)$ is an analytic function of τ^2, $D^n(\tau^{2n} h_n(\tau))$ is an even function of τ.

Equation (V.16) is a special case of the Prandtl integro-differential equation of airfoil theory and has been extensively studied. It is shown in [8] that (V.16) can be reduced to a regular Fredholm equation of second kind, and from this it is established that there exists a solution $\mu(t)$, with Hölder-continuous derivative, which satisfies (V.15). It follows then that (V.14), with μ chosen as the solution of (V.16), is indeed a solution of (P. III).

VI. Singularities at the Edge

It is intended that some numerical computations based on the preceding considerations will be presented at a later time. Here we discuss a different problem. One feature of two-dimensional problems is the existence of a fairly extensive theory of the singularities produced at the confluence of two analytic boundary conditions, the theory having been initiated by Lewy [*9*] and continued by Lehman [*10*]. Since we have reduced our problem to a two-dimensional one, it is natural to apply this theory and then translate back into original problem. This we do now following initially the pattern of [*9*].

Let us note at the outset that the function $m(\varrho)$ of Section V is analytic. For as remarked the $h_n(r)$ are analytic functions of r^2; hence in (V.13) we have

$$\tau D^n[\tau^{2n} h_n(\tau)] = \tau \sum_{m=0}^{\infty} a_m \tau^{2m}.$$

Since

$$\int_0^{\varrho} \tau^{2m+1}(\varrho^2-\tau^2)^{-\frac{1}{2}}\,d\tau = \varrho^{2m+1}\int_0^1 t^{k+1}(1-t^2)^{-\frac{1}{2}}\,dt = k_m\,\varrho^{2m+1}, \qquad k_m < \pi/2,$$

we have accordingly

$$m(\varrho) = \sum_{m=0}^{\infty} b_m\,\varrho^{2m}, \tag{VI.1}$$

a convergent series for all ϱ.

From (V.7) we have, writing $f(y) = w(0, y)$ for complex $y = \xi + i\eta$,

$$\bar{f}(\bar{y}) = -f(-\bar{y}), \qquad \bar{f}'(\bar{y}) = f'(-\bar{y}).$$

Thus we have

$$2\varphi_\xi(\xi,\eta) = f'(y) + \bar{f}'(\bar{y}) = f'(y) + f'(-\bar{y}),$$
$$2\varphi(\xi,\eta) = f(y) + \bar{f}(\bar{y}) = f(y) - f(-\bar{y}),$$

and condition (V.13) becomes

$$f'(y) + f'(y^*) + k[f(y) - f(y^*)] = 2\Phi(y) \quad \text{for} \quad \arg y = \pi/2, \quad |y| < +2. \tag{VI.2}$$

Here y^* denotes the point obtained from y by a counterclockwise circuit about the point $y = i$. If we set

$$F(y) = f(y) - \int_0^y \Phi(\varrho)\,d\varrho, \tag{VI.3}$$

we can rewrite (VI.2) in the form

$$F'(y) + F'(y^*) + k[F(y) - F(y^*)] = 0, \tag{VI.4}$$

again for $\arg y = \pi/2$.

While we have derived (VI.4) initially for $\arg y = \pi/2$, we may use it as a functional relation to continue $F(y)$ onto the portion $|i-y|<2$ of a Riemann surface with $y=i$ as a logarithmic branch point★. Observe that if $F(y)$ is given on the sector $-\pi/2<\arg(y-i)<+3\pi/2$ of this surface, equation (VI.4) can be considered as an ordinary differential equation for $F(y^*)$ and $3\pi/2<\arg(y^*-1)<7\pi/2$. By continuing in both directions $F(y)$ is continued analytically to all values of $\arg(y-i)$ on the Riemann surface, the equation (VI.4) holding whenever $|y-i|<2$.

It is proved in [*11*] that a solution of (VI.4) must have the form

$$\text{(VI.5)} \qquad F(y) = c + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} (y-i)^{m+\frac{1}{2}} [(y-i)\log(y-i)]^n, \qquad c \text{ a constant},$$

the series converging absolutely and uniformly for $|y-i| \leqq \varrho_0$, ϱ_0 any positive number less than 2. Consider now the representation (V.11). Recalling that $w(0,\zeta)=f(\zeta)$, we have from (VI.4) and (VI.1) that

$$w(0,\zeta) = F(\zeta) + \sum_{m=1}^{\infty} b_m \zeta^{2m+1}.$$

But

$$\int_{-r}^{r} (r^2-\varrho^2)^{n-\frac{1}{2}} (y+i\varrho)^{2m+1}\, d\varrho = \sum_{k=0}^{m} a_{mk}\, y^{2k+1} \int_{0}^{r} (r^2-\varrho^2)^{n-\frac{1}{2}} \varrho^{2n-2k}\, d\varrho$$
$$= r^n \sum_{k=0}^{m} \beta_{mk}\, y^{2k+1} r^{2n-2k}.$$

Hence we can write (V.11) in the form

$$\text{(VI.6)} \qquad r^n w(r,y) = 2c_n r^{-n} \int_{-r}^{+r} (r^2-\varrho^2)^{n-\frac{1}{2}} F(y+i\varrho)\, d\varrho + S(r,y)$$

where $S(r,y)$ is a power series in r and y. Now if we introduce the variables

$$\zeta = y + i\varrho, \qquad z = y + ir, \qquad \bar{z} = y - ir,$$

the integral on the right side of (VI.6) can be written as

$$\int_{-r}^{+r} (r^2-\varrho^2)^{n-\frac{1}{2}} F(y+i\varrho)\, d\varrho = \int_{\bar{z}}^{z} [(z-\zeta)(\zeta-\bar{z})]^{n-\frac{1}{2}} F(\zeta)\, d\zeta = \tilde{I}(z,\bar{z}),$$

the integration being along any path joining $\bar{z}$ to z without crossing the slit, $\operatorname{Re}\zeta=0$, $-1<\operatorname{Im}\zeta<+1$. With the correct choice of path we have

$$\tilde{I}(z,\bar{z}) = 2\operatorname{Re}\int_{0}^{z} [(z-\zeta)(\zeta-\bar{z})]^{n-\frac{1}{2}} F(\zeta)\, d\zeta = 2\operatorname{Re} I_1(z,\bar{z})$$

along a path joining 0 and z without crossing the slit. Setting $\xi=\zeta-i$, $\eta=z-i$, we have

$$\text{(VI.7)} \qquad I_1 = \int_{-i}^{\eta} [(\eta-\xi)(\xi-\bar{\eta}+2i)]^{n-\frac{1}{2}} F(\xi+i)\, d\xi.$$

Suppose now that $|\eta|<1$ and that we choose for the path of integration in I_1 a straight line joining $\xi=-i$ to $\xi=\eta$. Then for ξ on this path we have $|\xi|<|\eta-2i|$;

★ In this relation y^* of course continues to have the same meaning as above; that is, it is an image of y on the Riemann surface.

hence
$$(\xi - \bar{\eta} + i)^{n-\frac{1}{2}} = \sum_{k=0}^{\infty} \gamma_{nk}\, \xi^k (\bar{\eta} - 2i)^{-k},$$
or, since $|\eta| < 1$,

(VI.8)
$$(\xi - \bar{\eta} + i)^{n-\frac{1}{2}} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \gamma_{nkm}\, \xi^k \bar{\eta}^m.$$

If we enter the expressions (VI.8) and (VI.5) into (VI.7), we find
$$I_1 = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} B_{rs}(\bar{\eta}) \int_{-i} (\eta - \xi)^{n-\frac{1}{2}} \xi^{r+s+\frac{1}{2}} (\log \xi)^s \, d\xi$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} B_{rs}(\bar{\eta})\, \eta^{n+r+s+\frac{1}{2}} \int_{-i}^{1} (1-\tau)^{n-\frac{1}{2}} \tau^{r+s+\frac{1}{2}} (\log \tau + \log \eta)^s \, d\tau,$$
the coefficients B_{rs} being analytic functions of $\bar{\eta}$. From this it follows with a little calculation that
$$I_1 = \eta^n \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} b_{kmj}\, \eta^k \bar{\eta}^m (\eta \log \eta)^j$$
if $|\eta| < 1$. Thus, if we collect our results, we find that

(VI.9)
$$r^n w(r, y) = \operatorname{Re} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} a_{kmj} (z-i)^k (z+i)^m [(z-i)\log(z-i)]^j, \quad z = y + i r,$$
the series converging for $|z - i| < 1$. Note that the function $S(r, y)$ of equation (VI.6) has been incorporated into the series (VI.9).

Let us now recall the role played in the solution of (P. I) by solutions of (P. II). We first found a finite number N of solutions of (P. II) having the form $v^n(r, y) \cos n\vartheta$. Then for a suitable choice of constants A_i the function
$$f(r, \vartheta) = \sum_{n\;0}^{N} A_n v_y^n(r, 0) \cos n\vartheta$$
was found to be equal to
$$U_y(r, \vartheta, 0) - k\, U(r, \vartheta, 0),$$
where U is a solution of (P. I) for the right side of (C) equal to $G^N(r, \vartheta)$. Since each function $v^n(r, y)$ is of the form $r^n w(r, y)$, where w is a function of the type we have studied in this section, we conclude from (VI.9) that
$$v_y^n(r, 0) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} P_{km} (1-r)^k [(1-r)\log(1-r)]^m.$$
Thus we have

(VI.10)
$$U_y(r, \vartheta, 0) - k\, U(r, \vartheta, 0) = \sum_{n=0}^{N} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} Q_{km}^n (1-r)^k [(1-r)\log(1-r)]^m \cos n\vartheta.$$

Finally we remark that in this solution $U_y(r, \vartheta, 0)$ equals $G^N(r, \vartheta)$, and
$$G^N(r, \vartheta) = \sum_{n=0}^{N} \left(\sum_{k=0}^{\infty} c_k (1-r)^k \right) \cos n\vartheta.$$
It follows that the function $U(r, \vartheta, 0)$, which except for a numerical factor is the fluid pressure on the disk, has a development of the same form as (VI.10). Here the singularity at the edge has exactly the same form as that obtained in [*9*] for the two-dimensional problem.

Appendix

The purpose of this Appendix is to study the coefficients in the linear equations (III.10). At first glance these equations would appear to present difficulties since they require differentiations of the singular operator U. We shall show, however, that these difficulties are only illusory.

To begin with we remark that the kernel $G(r, \vartheta, 0, \varrho, \psi)$ of $U(r, \vartheta, 0; f)$ can be evaluated explicitly. By [12] (p. 196) we have

$$\int_0^\infty \frac{e^{-k\eta}}{[\eta^2+\mu^2]^{\frac{1}{2}}}\, d = \frac{\pi}{2}\left(S_0(k\mu) - Y_0(k\mu)\right)$$

where Y_0 is the singular Bessel function of first kind and S_0 is the Struve function. Thus by (II.2) we can write

$$\text{(A.1)}\qquad \begin{aligned} G(r, \vartheta, 0, \varrho, \psi) = B(\mu) &= \frac{1}{\mu} - \frac{k\pi}{2}\left[S_0(k\mu) + Y_0(k\mu) + i J_0(k\mu)\right], \\ \mu^2 &= r^2 + \varrho^2 - 2r\varrho\cos(\vartheta - \psi). \end{aligned}$$

Let us now study the form of the functions $f(r, \vartheta)$, $f_n(r, \vartheta)$ appearing in (IV.10). As remarked in Section V they must all have the form $r^n F(r)\cos n\vartheta$. Thus in (III.10) we are concerned with functions of the form,

$$\text{(A.2)}\qquad \begin{aligned} \varkappa(r, \vartheta) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \varrho^{n+1} F(\varrho)\cos n\psi\, B(\mu)\, d\varrho\, d\psi \\ &= \frac{1}{2\pi}\left(\int_0^{2\pi}\int_0^1 \varrho^{n+1} F(\varrho)\cos n\alpha\, A(\tau)\, d\varrho\, d\alpha\right)\cos n\vartheta; \\ \tau^2 &= r^2 + \varrho^2 - 2r\varrho\cos\alpha, \qquad A(\tau) = B(\tau^{\frac{1}{2}}), \end{aligned}$$

and the problem is to differentiate such functions with respect to r. We obtain the desired result from the following relation, easily verified by induction:

$$\frac{\partial^n}{\partial r^n} B(\tau) = \sum_{j=0}^{[n/2]} c_j^n\, B^{(n-j)}\, \tau_r^{n-j}\, \tau_{rr}^j, \qquad c_0^n = 1.$$

Then from (A.2) we have

$$\frac{\partial^n \varkappa(r, \vartheta)}{\partial r^n} = \frac{1}{2\pi}\left\{\sum_{j=0}^{[n/2]} c_j^n \int_0^{2\pi}\int_0^1 \varrho^{n+1} F(\varrho)\cos n\psi\, B^{(n-j)}(2r - 2\varrho\cos\psi)^{n-2j}\cdot 2^j\, d\varrho\, d\psi\right\}\cos n\vartheta$$

or

$$\frac{\partial^n \varkappa(0, \vartheta)}{\partial r^n} = \frac{1}{2\pi}\sum_{j=0}^{[n/2]} 2^{n-j}\int_0^{2\pi}\int_0^1 \varrho^{2n-2j+1} F(\varrho)\, B^{n-j}(\varrho^2)\, d\varrho\, d\psi.$$

But

$$\cos^m\psi = \frac{1}{2^{m-1}}\cos m\psi + a_1\cos(m-2)\psi + \cdots,$$

so that the only term remaining after the ψ integration is that for $j=0$, and

$$\text{(A.3)}\qquad \frac{\partial^n \varkappa(0, \vartheta)}{\partial r^n} = \left(\int_0^1 {}^{2n+1}F(\varrho)\, B^{(n)}(\varrho^2)\, d\varrho\right)\cos n\vartheta.$$

Now the Struve function $S_0(z)$ is an analytic function of z, while $Y_0(z)$ equals $2\pi^{-1} J_0(z) \log z$ plus an analytic function of z^2. Hence by (A.1) we have

$$A(\tau) = \frac{1}{\tau} + \left(\sum_{j=0}^{\infty} a_j \tau^j\right) \log \tau + \sum_{j=0}^{\infty} b_j \tau^j + \tau^{\frac{1}{2}} \sum_{j=0}^{\infty} c_j \tau^j,$$

or

$$A^{(n)}(\varrho^2) = \frac{\gamma}{\varrho^{2n+1}} + \sum_{j=-n}^{\infty} \alpha_j \varrho^{-2j} + \sum_{j=0}^{\infty} \beta_j \varrho^{2j} \log \varrho + \sum_{j=-n}^{\infty} \gamma_j \varrho^{-2j+1}.$$

It follows that the integrals (A.3) are all convergent.

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Diffraction by a Quarter-Plane

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1. Introduction

We report on our study [*6*] of the diffraction by a quarter-plane. This problem differs from its direct ancestor the Poincaré-Sommerfeld half-plane problem in being essentially three-dimensional in character. It has been known for some time as an open problem of diffraction theory.

Our analysis is in essence a generalization of the function-theoretical method of WIENER & HOPF [*9*] from one to two complex variables. Relevant notations of two-variable function theory and the two-dimensional Laplace transformation are given in Section 2. We state the physical problem to be considered, together with the corresponding boundary problem, in Section 3. Next, in Section 4, we show that the boundary problem is formally equivalent to a transform equation relating two known and four unknown two-variable functions.

The solution of the transform equation requires a certain two-variable factorization. This part of the analysis (Section 5) directly generalizes the basic factorization process of the one-variable WIENER-HOPF theory. In Section 6, the unknown functions are expressed in terms of the factors obtained in Section 5, and it is verified that the conditions of Section 4 are met, *i.e.* the transform equation is satisfied, and the functions involved possess certain required analyticity and integrability properties. We remark that the relationship between the factorization of Section 5 and the solution given in Section 6 is less straightforward than the corresponding relationship in one dimension.

We summarize the mathematical argument, in Section 7, by proving a theorem which asserts that the solution of the transform equation given in Section 6 corresponds to an actual (*i.e.*, not merely a formal) solution of our boundary problem on the quarter-plane. Physical properties of the diffracted field are deduced in Section 8: we show that this field is an outgoing wave at infinity, and discuss its behavior near the edges and the corner of the diffracting screen.

2. Notations

We shall work in a Euclidean space E_4: the schlicht finite s-space, where $s=\sigma+i\tau$ and $\sigma=(\sigma_1,\sigma_2)$, $\tau=(\tau_1,\tau_2)$ are real points of E_2. The L_2 norm of an analytic function $\varphi(s)=\varphi(s_1,s_2)$ is defined by

$$\|\varphi(s)\|_2=\left\{\iint_{-\infty}^{\infty}|\varphi(\sigma+i\tau)|^2\,d\tau_1\,d\tau_2\right\}^{\frac{1}{2}} \tag{2.1}$$

and is in general bounded only for restricted domains of σ. A typical domain of boundedness is a tube (see [2]): a tube $T(\Delta)$ in s-space is a point set

$$\sigma_j \in \Delta, \qquad -\infty < \tau_j < +\infty \tag{2.2}$$

($j=1, 2$) where Δ, called the basis of the tube, is any set in the $\sigma_1 \sigma_2$-plane. For brevity, we agree that in this paper the subscript j is always to run over the values 1, 2, as in (2.2).

If a function $\varphi(s)$ is analytic and of bounded L_2 norm in a tube with basis

$$D: \quad \gamma_j \leqq \sigma_j \leqq \delta_j, \tag{2.3}$$

then by a theorem of BOCHNER (see [2], p. 738) it can be written as a sum of four functions $\varphi_n(s)$, analytic and bounded in the respective octant-shaped tubes, or quarter-spaces, $T(D, n)$, with

$$D,1: \quad \sigma_1 > \gamma_1, \qquad \sigma_2 > \gamma_2, \tag{2.4.1}$$

$$D,2: \quad \sigma_1 < \delta_1, \qquad \sigma_2 > \gamma_2, \tag{2.4.2}$$

$$D,3: \quad \sigma_1 < \delta_1, \qquad \sigma_2 < \delta_2, \tag{2.4.3}$$

$$D,4: \quad \sigma_1 > \gamma_1, \qquad \sigma_2 < \delta_2. \tag{2.4.4}$$

The functions $\varphi_n(s)$ are uniquely determined up to additive constants. We note that the subscript n in this paper will (except when otherwise specified) always run over the values 1, 2, 3, 4.

Consider a function, say $V(s)\, W(s)$, analytic and of bounded L_2 norm in a tube $T(D)$ as above. This function is by BOCHNER'S theorem uniquely decomposable into a sum of four functions $A_n(s)$, analytic and bounded in the respective tubes $T(D, n)$; we shall denote

$$[V(s)\, W(s)]_n = A_n(s), \tag{2.5}$$

$$[V(s)\, W(s)]_{\sigma_1^+} = A_1(s) + A_4(s), \tag{2.6}$$

$$[V(s)\, W(s)]_{\sigma_2^+} = A_1(s) + A_2(s). \tag{2.7}$$

The indicated decompositions are carried out by means of CAUCHY'S formula, in two variables or one, as follows. Let Z_1, Z_2 denote vertical contours from $-i\infty$ to $+i\infty$; then we have, *e.g.*

$$A_1(s) = \frac{1}{(2\pi i)^2} \int\limits_{Z_1} \int\limits_{Z_2} \frac{V(z)\, W(z)\, dz_1\, dz_2}{(s_1 - z_1)(s_2 - z_2)} \tag{2.8}$$

for $\sigma_j > \mathrm{Re}(z_j) > \gamma_j$ (see (2.4.1)). Also,

$$A_1(s) + A_4(s) = \frac{1}{2\pi i} \int\limits_{Z_1} \frac{V(z_1, s_2)\, W(z_1, s_2)\, dz_1}{s_1 - z_1} \tag{2.9}$$

for $\sigma_1 > \mathrm{Re}(z_1) > \gamma_1$, and

$$A_1(s) + A_2(s) = \frac{1}{2\pi i} \int\limits_{Z_2} \frac{V(s_1, z_2)\, W(s_1, z_2)\, dz_2}{s_2 - z_2} \tag{2.10}$$

for $\sigma_2 > \mathrm{Re}(z_2) > \gamma_2$.

We denote by $\mathscr{L}$ the two-dimensional Laplace transform (see [3]): given a function $h(x_1, x_2)$ of a suitable class, we define the Laplace transform $H(s)$ by

$$H(s) = \mathscr{L}[h(x)] = \iint\limits_{-\infty}^{\infty} h(x) \exp(-s \cdot x)\, d x_1\, d x_2$$

with $s \cdot x = s_1 x_1 + s_2 x_2$, $s_j = \sigma_j + i \tau_j$ as before. A tube, as defined above, is clearly a typical domain of convergence for the integral on the right, and therefore a typical analyticity domain for a Laplace transform $H(s)$. The inverse transformation is defined by

$$h(x) = \mathscr{L}^{-1}[H(s)] = \frac{1}{(2\pi i)^2} \int\limits_{S_1} \int\limits_{S_2} H(s) \exp(s \cdot x)\, d s_1\, d s_2$$

where S_1, S_2 denote vertical contours from $-i\infty$ to $+i\infty$.

3. Formulation of the problem

We consider the diffraction of an E-polarized harmonic plane wave by a perfectly conducting screen which occupies the quarter-plane $x_1 \geqq 0$, $x_2 \geqq 0$, $x_3 = 0$. The incident plane wave is

$$u_0(x_1, x_2, x_3) = \exp(-a_1 x_1 - a_2 x_2 - a_3 x_3), \tag{3.1}$$

where

$$a_1 = i k \sin\vartheta_0 \cos\varphi_0, \qquad a_2 = i k \cos\vartheta_0, \qquad a_3 = i k \cos\vartheta_0 \sin\varphi_0$$

with $k = p - iq$ $(p > 0,\ q > 0)$ and $0 < \vartheta_0 \leqq \pi/2$, $0 < \varphi_0 \leqq \pi/2$. Our problem may alternately be taken to refer to the diffraction of plane sound waves, with velocity potential u_0, by a perfectly reflecting quarter-infinite screen.

Let the total field be $u_0 + u$; then the scattered field u is a three-dimensional wave function which cancels u_0 on the screen. Next, denote the surface current density by $f(x) = f(x_1, x_2)$:

$$f(x) = \left.\frac{\partial u}{\partial x_3}\right|_{x_3 = 0+} - \left.\frac{\partial u}{\partial x_3}\right|_{x_3 = 0-}. \tag{3.2}$$

It is known that $f(x)$ must vanish off the screen, *i.e.*, for $E[x_1 < 0 \cup x_2 < 0]$, and we also ask that $f(x)$ have the behavior of $u_0(x_1, x_2, 0)$ at infinity.

Two more conditions on u are needed. Our assumption that k is complex means (see the discussion in [1], pp. 154—55) that SOMMERFELD'S radiation condition may be replaced by the condition that u be outgoing at infinity, *i.e.*, that it have the behavior of the exponentially damped free-space GREEN'S function $(e^{-ikR})/(4\pi R)$, for large values of $R = +(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. Also, to ensure a finite flow of energy into the scattering region, we require that $u(x_1, x_2, 0)$ be of integrable square (as a function of two variables) at the origin.

Finally, we shall in the interests of uniqueness ask that $f(x)$ be of integrable square at the origin. This condition is based on our assumption that the current density, in a physically acceptable solution, has the behavior $f \sim r^{-1+\delta}$ (with $r^2 = x_1^2 + x_2^2$, $0 < \delta < \frac{1}{2}$) near $r = 0$.

We now have the following mixed boundary problem on the quarter-plane to be solved for the scattered field $u(x_1, x_2, x_3)$:

$$(\Delta_{x_1 x_2 x_3} + k^2)\, u = 0; \tag{3.3}$$

$$u(x_1, x_2, 0) = -e^{-a_1 x_1 - a_2 x_2}, \quad \text{for } x \geqq 0; \tag{3.4}$$

$$f(x) = 0, \quad \text{for } E[x_1 < 0 \cup x_2 < 0]; \tag{3.5}$$

$$f(x) = O(e^{-a_1 x_1 - a_2 x_2}), \quad \text{for large } x_j > 0; \tag{3.6}$$

$$u = O(e^{-qR}/R), \quad \text{for large } R, \tag{3.7}$$

provided $x_3 \neq 0$, while if $x_3 = 0$,

$$u(x_1, x_2, 0) = O\left(e^{-q|x|}/|x|^{\frac{1}{2}}\right) \tag{3.7.1}$$

for large values of $|x| = +(x_1^2 + x_2^2)^{\frac{1}{2}}$, with $x = (x_1, x_2)$ any point in $E[x_1 < 0 \cup x_2 < 0]$;

$$u(x_1, x_2, 0) \in L_2\left(|x_j| \leqq d_j\right) \tag{3.8}$$

for any finite d_1, d_2;

$$f(x) \in L_2(0 \leqq x_j \leqq c_j) \tag{3.9}$$

for any finite c_1, c_2.

Conditions (3.3) to (3.9) completely describe our problem. The physically (and mathematically) reasonable assumption that the field scattered by the quarter-infinite screen $x_1 \geqq 0$, $x_2 \geqq 0$, $x_3 = 0$ is indistinguishable for large positive x_1 from that scattered by the semi-infinite screen

$$\Sigma_1: \quad -\infty < x_1 < +\infty, \quad x_2 \geqq 0, \quad x_3 = 0 \tag{3.10}$$

and indistinguishable for large positive x_2 from that scattered by

$$\Sigma_2: \quad x_1 \geqq 0, \quad -\infty < x_2 < +\infty, \quad x_3 = 0 \tag{3.11}$$

is not a logically independent requirement. The solution derived in Sections 4, 5, 6, and proved in Section 7 to meet conditions (3.3) to (3.9), will be shown in Section 8 to correspond to a field which reduces, for large positive x_1 (x_2), to that scattered by Σ_1 (Σ_2).

4. The transform equation

A formal solution of (3.3) which is outgoing at infinity, *i.e.*, satisfies (3.7), is

$$u(x_1, x_2, x_3) = \frac{1}{(2\pi i)^2} \int_{S_1}\int_{S_2} F(s)\, K(s) \exp\left[s \cdot x + \frac{|x_3|}{2K(s)}\right] ds_1\, ds_2 \tag{4.1}$$

where $s \cdot x = s_1 x_1 + s_2 x_2$; S_1, S_2 denote vertical contours from $-i\infty$ to $+i\infty$; $F(s) = F(s_1, s_2)$ is an unknown function; and

$$K(s) = (i/2)\,(s_1^2 + s_2^2 + k^2)^{-\frac{1}{2}} \tag{4.2}$$

with the branch of the square root determined by the requirement that it reduce to $+k$ on the manifold $s_1^2 + s_2^2 = 0$. From (4.1), we have

$$u(x_1, x_2, 0) = \frac{1}{(2\pi i)^2} \int_{S_1}\int_{S_2} F(s)\, K(s) \exp(s \cdot x)\, ds_1\, ds_2, \tag{4.3}$$

$$f(x_1, x_2) = \frac{1}{(2\pi i)^2} \int_{S_1}\int_{S_2} F(s) \exp(s \cdot x)\, ds_1\, ds_2. \tag{4.4}$$

Now (4.4) defines $f(x)$ as the inverse Laplace transform of $F(s)$. We shall, correspondingly, define $F(s)$ as the Laplace transform of $f(x)$. From (3.5), we then have

$$F(s)=\int_0^\infty\int_0^\infty f(x_1,x_2)\exp(-s\cdot x)\,dx_1\,dx_2 \tag{4.5}$$

and may show (using (3.6), (3.9), and the triangle inequality) that $f(x)\exp(-s\cdot x)$ is of bounded L_2 norm, for $\sigma_j>-\mathrm{Re}(a_j)$. It follows from PLANCHEREL'S theorem (see [3], Prop. 4.1) that $F(s)$ is analytic in the quarter-space $T(\gamma,1)$ with basis

$$\gamma,1:\quad \sigma_j>-\mathrm{Re}(a_j) \tag{4.6}$$

and of uniformly bounded L_2 norm for $\sigma\in(\gamma,1)$.

Next, we see from (4.2) that $K(s)$ is analytic and uniformly bounded in the tube $T(B)$ with basis

$$B:\quad |\sigma_j|\leqq b_j \tag{4.7}$$

where

$$b_j>0,\qquad (b_1^2+b_2^2)^{\frac{1}{2}}<q=|\mathrm{Im}\,k|. \tag{4.7.1}$$

Since the intersection $B\cap(\gamma,1)$ is nonempty, we conclude that $F(s)\,K(s)$ is analytic in the tube with basis $B\cap(\gamma,1)$ and of uniformly bounded L_2 norm for $\sigma\in[B\cap(\gamma,1)]$. From BOCHNER'S theorem (see Section 2), we have a unique additive decomposition

$$F(s)\,K(s)=G_1(s)+G_2(s)+G_3(s)+G_4(s) \tag{4.8}$$

with $G_n(s)$ analytic and bounded in a quarter-space $T(\delta,n)$, where the basis (δ,n) covers the n^{th} quadrant of the $\sigma_1\,\sigma_2$-plane and also contains the interior of $B\cap(\gamma,1)$.

Notice now that (4.3) defines $u(x,0)=u(x_1,x_2,0)$ as the inverse Laplace transform of (FK); alternatively, we have

$$F(s)\,K(s)=\mathscr{L}[u(x,0)], \tag{4.9}$$

and we see, from (3.4), (3.7.1), (3.8), that the right side of (4.9) is a function of bounded L_2 norm for $\sigma\in[B\cap(\gamma,1)]$, which is in agreement with our result for (FK). Let us write

$$u(x,0)=\sum_{n=1}^{4}u_n(x,0) \tag{4.10}$$

where $u_n(x,0)$ vanishes outside the n^{th} quadrant of the x_1x_2-plane; *i.e.*, if $Y(x_j)$ denotes Heaviside's function $(Y(x_j)=0,\ x_j<0;\ Y(x_j)=1,\ x_j>0)$, define $u_1(x,0)=u(x,0)\,Y(x_1)\,Y(x_2)$, $u_2(x,0)=u(x,0)\,Y(-x_1)\,Y(x_2)$, $u_3(x,0)=u(x,0)\,Y(-x_1)\,Y(-x_2)$, $u_4(u,0)=u(x,0)\,Y(x_1)\,Y(-x_2)$. Then, from (4.8), we have

$$G_n(s)=\mathscr{L}[u_n(x,0)], \tag{4.11}$$

and, in particular, from (3.4)

$$G_1(s)=-(s_1+a_1)^{-1}(s_2+a_2)^{-1}. \tag{4.11.1}$$

We have now applied all the conditions of our boundary problem and may summarize the results as follows. The right side of (4.1) represents a function

$u(x_1, x_2, x_3)$ which satisfies Eqs. (3.3) to (3.9), provided that $F(s)$ is analytic in $T(\gamma, 1)$ and of uniformly bounded L_2 norm for $\sigma \in (\gamma, 1)$ (see (4.6)), and that

$$F(s)\,K(s) = G_1(s) + G_2(s) + G_3(s) + G_4(s) \tag{4.12}$$

where the known functions $K(s)$, $G_1(s)$ are given by (4.2), (4.11.1); and the unknown functions G_2, G_3, G_4 are according to (4.11) related to $u(x, 0)$ for $E[x_1 < 0 \cup x_2 < 0]$. Our problem therefore consists of solving the transform equation (4.12) for the unknown functions F, G_2, G_3, G_4; it is understood these solutions must be analytic and of uniformly bounded L_2 norm in the respective domains $\sigma \in (\gamma, 1)$ and $\sigma \in (\delta, n)$ for $n = 2, 3, 4$. F is required to be analytic in $T(\gamma, 1)$ in order that condition (3.5) be met.

We may instructively compare the present problem with the simpler one we considered in [7]: a vector problem of diffraction by a semi-infinite screen was there reduced, by the introduction of double Laplace transforms, to the problem of solving a transform equation (see [7], Eq. (2.5)) whose form was precisely that of (4.12). There were, however, two decisive simplifications, whose effect may be understood by our supposing, for the moment, that (4.12) is to be solved under the following conditions: (1) the known functions include not only K, G_1, but also G_4; (2) F is required to be analytic only for $\sigma_1 > -\alpha_1$, $|\sigma_2| < \alpha_2$, where $\alpha_1 \geqq 0$, $\alpha_2 > 0$. This simplified problem may be solved in the manner of [7], namely, on the basis of the factorization

$$K(s) = (i/2)\,M_+(s)\,M_-(S) \tag{4.13}$$

with

$$M_+(s) = [s_1 + i(s_2^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}}, \tag{4.14}$$

$$M_-(s) = [s_1 - i(s_2^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}}. \tag{4.15}$$

Let $k = p - iq$ as before, and make the choice

$$(s_2^2 + k^2)^{\frac{1}{2}}\big|_{s_2=0} = +k;$$

then the factors $M_+(s)$, $M_-(s)$ are analytic (and nonzero), respectively, for $\sigma_1 \geqq 0$, $|\sigma_2| < q$; $\sigma_1 \leqq 0$, $|\sigma_2| < q$. We solve (4.12), under the simplified conditions assumed, by arguments of the type given in [7], with the results:

$$F(s) = \frac{2i}{M_+(s)} \left[\frac{G_1(s) + G_4(s)}{M_-(s)}\right]_{\sigma_1^+}, \tag{4.16}$$

$$G_2(s) + G_3(s) = -M_-(s) \left[\frac{G_1(s) + G_4(s)}{M_-(s)}\right]_{\sigma_1^-}, \tag{4.17}$$

where the bracket-subscript notation is that of (2.6); it is understood that (2.6) has the counterpart

$$[V(s)\,W(s)]_{\sigma_1^-} = A_2(s) + A_3(s).$$

Notice that the right side of (4.16) is analytic, as required, for $\sigma_1 \geqq 0$, $|\sigma_2| < \alpha_2$.

We compare the simplified problem solved by (4.16), (4.17) with the problem we have actually to consider by first observing that we have more information in the simplified problem, *i.e.*, we are given

$$[F(s)\,K(s)]_{\sigma_1^+} = G_1(s) + G_4(s) \tag{4.18}$$

with both G_1, G_4 known, while in our actual problem we have only

$$[F(s)\,K(s)]_1 = G_1(s)\,. \tag{4.19}$$

Also, the simplified problem asks, in effect, that the complete additive decomposition of $F(s)$ have the form

$$F(s) = F_1(s) + F_4(s)\,, \tag{4.20}$$

while our actual problem makes the more stringent demand that $F(s)$ have the complete decomposition

$$F(s) = F_1(s)\,. \tag{4.21}$$

5. Factorization

Our solution is based on a two-variable factorization of $K(s)$ (see (4.2)):

$$K(s) = (i/2) \prod_{n=1}^{4} M_n(s) \tag{5.1}$$

with the $M_n(s)$ analytic and nonvanishing in respective octant-shaped tubes (quarter-spaces) $T(B, n)$ whose bases (B, n) have B as intersection (see (4.7) for B). The (B, n) are given by

$$(B,1):\quad \sigma_1 > -b_1, \qquad \sigma_2 > -b_2, \tag{5.2.1}$$

$$(B,2):\quad \sigma_1 < b_1, \qquad \sigma_2 > -b_2, \tag{5.2.2}$$

$$(B,3):\quad \sigma_1 < b_1, \qquad \sigma_2 < b_2, \tag{5.2.3}$$

$$(B,4):\quad \sigma_1 > -b_1, \qquad \sigma_2 < b_2. \tag{5.2.4}$$

The problem of deriving the factorization (5.1) generalizes the classical one-variable problem of Wiener & Hopf [9]. As in the one-variable case, we may take our problem to be that of additively decomposing $\log K(s)$.

The additive decomposition problem is solved as follows. Let us allow the parameter k to vary over a disk η_k with center $k = (p, -q)$ and radius η. From the analyticity of $K(s)$, η may be so chosen that $K(s)$ is an analytic function of s, k throughout η_k. It is then permissible to differentiate $K(s)$ with respect to k ($k \in \eta_k$), and to form the logarithmic derivative (the prime denotes differentiation with respect to k):

$$\frac{K'(s)}{K(s)} = \frac{-k}{s_1^2 + s_2^2 + k^2} \tag{5.3}$$

whose analyticity domain $T(B)$ is the same as that of $K(s)$, and which we shall take as the function to be decomposed additively. The decomposition is to have the form

$$\frac{K'(s)}{K(s)} = \sum_{n=1}^{4} \frac{M_n'(s)}{M_n(s)} \tag{5.4}$$

with the n^{th} term on the right analytic in $T(B, n)$. Given (5.4), we would expect to be able to justify an integration with respect to k, with the result

$$\log K(s) = \sum_{n=1}^{4} \log M_n(s)\,, \tag{5.5}$$

which is equivalent (after constants of integration are properly accounted for) to (5.1).

We begin the calculation by observing that (5.3) may be rewritten as

$$\frac{K'(s)}{K(s)} = \frac{-ik}{4}\,\mathscr{L}\left[H_0^{(2)}(k|x|)\right] \tag{5.6}$$

with $|x| = +(x_1^2 + x_2^2)^{\frac{1}{2}}$; both sides of the equation are analytic in $T(B)$. To make use of (5.6) in deriving the required result of the form (5.4), we express $\mathscr{L}$ as the sum of its restrictions to the four quadrants of the $x_1 x_2$-plane. We have

$$\mathscr{L} = \sum_{n=1}^{4} \mathscr{L}_n \tag{5.7}$$

where

$$\mathscr{L}_n = \iint_{-\infty}^{\infty} dx_1\, dx_2\, Y(\varepsilon_{n1} x_1)\, Y(\varepsilon_{n2} x_2) \exp(-s \cdot x) \tag{5.8}$$

and

$$\begin{aligned} \varepsilon_{n1} &= +1 \qquad (n = 1, 4), \\ &= -1 \qquad (n = 2, 3), \end{aligned} \tag{5.9.1}$$

$$\begin{aligned} \varepsilon_{n2} &= +1 \qquad (n = 1, 2), \\ &= -1 \qquad (n = 3, 4). \end{aligned} \tag{5.9.2}$$

Y denotes Heaviside's function, defined for any real argument t by

$$Y(t) = 0, \qquad t < 0; \qquad Y(t) = 1, \qquad t > 0.$$

It then follows from PARSEVAL'S theorem, *i.e.* the result (see [3])

$$\mathscr{L}[g f] = \mathscr{L}[g] \mathbin{\#} \mathscr{L}[f], \tag{5.9.3}$$

where $\#$ denotes convolution, that the operations $\mathscr{L}_n$ map $H_0^{(2)}(k|x|)$ into functions analytic in the octant-shaped tubes $T(B, n)$. The required additive decomposition is therefore given by

$$\frac{K'(s)}{K(s)} = \frac{-ik}{4} \sum_{n=1}^{4} \mathscr{L}_n\left[H_0^{(2)}(k|x|)\right]. \tag{5.10}$$

The first term in the summation on the right side of (5.10) is the known transform (see [8], Eq. (3.5))

$$\mathscr{L}_1\left[H_0^{(2)}(k|x|)\right] = \Lambda(s_1, s_2) = -4K^2(s)\left[-i + s_1 P_2 + s_2 P_1\right] \tag{5.11}$$

with

$$P_j = t_j^{-1}\{1 + (2i/\pi)\log[(1/k)(s_j + t_j)]\} \tag{5.11.1}$$

and

$$t_j = (s_j^2 + k^2)^{\frac{1}{2}}. \tag{5.11.2}$$

Next, each of the remaining three transforms on the right side of (5.10) may be reduced to an integral over the first quadrant of the $x_1 x_2$-plane by changing the sign of the s-component corresponding to the negative x-component. Our result, in the notations of (5.9.1), (5.9.2), is

$$\mathscr{L}_n\left[H_0^{(2)}(k|x|)\right] = \Lambda(\varepsilon_{n1} s_1, \varepsilon_{n2} s_2) \tag{5.12}$$

with $A(s)$ defined by (5.11). The decomposition (5.10), of the desired form (5.4), is now explicit.

Integrating (5.10) with respect to k, we have

$$\log K(s) = \frac{-i}{4} \sum_{n=1}^{4} \int^{k} k\, \mathscr{L}_n [H_0^{(2)}(k|x|)]\, dk + \alpha_0(s) \tag{5.13}$$

where $\alpha_0(s)$ is a constant of integration (*i.e.*, independent of k). From (4.2),

$$\log K(s) = (i\pi/2) - \log 2 - (\tfrac{1}{2}) \log(s_1^2 + s_2^2 + k^2) \tag{5.14}$$

in $T(B)$. Comparing (5.13), (5.14), we infer that $\alpha_0(s)$ is everywhere equal to $(i\pi/2) - \log 2$, and the factorization

$$K(s) = (i/2) \prod_{n=1}^{4} \exp\left\{(-i/4) \int^{k} k\, \mathscr{L}_n [H_0^{(2)}(k|x|)]\, dk\right\} \tag{5.15}$$

of the desired form (5.1) follows from (5.13). The functions $M_n(s)$ of (5.1) are then given by

$$M_n(s) = \exp\left\{(-i/4) \int^{k} k\, \mathscr{L}_n [H_0^{(2)}(k|x|)]\, dk\right\} \tag{5.16}$$

(see (5.11), (5.12)) and, in accordance with our argument, are based on Parseval's theorem, analytic in the octant-shaped tubes $T(B, n)$; they are, as required, nonvanishing. Observe also that we have (with $M_+(s)$, $M_-(s)$ defined by (4.14), (4.15))

$$M_1(s)\, M_4(s) = M_+(s)\,, \tag{5.17}$$

$$M_2(s)\, M_3(s) = M_-(s)\,. \tag{5.18}$$

If we denote

$$M^+(s) = [s_2 + i(s_1^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}}, \tag{5.19}$$

$$M^-(s) = [s_2 - i(s_1^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}}, \tag{5.20}$$

we have, further,

$$M_1(s)\, M_2(s) = M^+(s)\,, \tag{5.21}$$

$$M_3(s)\, M_4(s) = M^-(s)\,. \tag{5.22}$$

Growth estimates for the $M_n(s)$ may be derived by carrying out the integration on the right side of (5.16). The results obtained are

$$M_n(s) \sim (s_1^2 + s_2^2)^{-\frac{1}{8}} \tag{5.23}$$

for large $|s_1^2 + s_2^2|$, with $s \in T(B, n)$.

6. Solution of the transform equation; verification

The transform equation (4.12) poses the problem of determining the four unknown functions F, G_2, G_3, G_4 from the information that

$$[F(s)\, K(s)]_1 = G_1(s)\,, \tag{6.1}$$

i.e., from the transform equivalent of the physical condition (3.4). We also require that $F(s)$ be analytic in $T(\gamma, 1)$, and of bounded L_2 norm for $\sigma \in (\gamma, 1)$,

so that its additive decomposition is given in full by

$$F(s) = F_1(s), \tag{6.2}$$

which is the equivalent of the physical condition (3.5).

We assert that a function $F(s)$ which satisfies (6.1), (6.2), and which corresponds, by means of (4.1), to a solution of the boundary problem stated by Eqs. (3.3) to (3.9), is given by

$$F(s) = \frac{-2i}{M_1(s)}\left[\frac{P_1(s)}{M_2(s)}\right]_{\sigma_1^+} \tag{6.3}$$

with

$$P_1(s) = \left[\frac{G_1(s)\, M_2(s)\, M_3(s_1, -a_2)}{M_2(-a_1, s_2)\, M_3(-a_1, -a_2)\, M_3(s)\, M_4(s)}\right]_1 \tag{6.4}$$

where the notation is that of (2.5), (2.6). This means, according to (2.8), (2.9), that the expressions on the right sides of (6.3), (6.4) are explicit integrals. The assertion we make in (6.3), (6.4) is then that

$$F(s) = \frac{[-2i/M_1(s)]}{2\pi i}\int\limits_{Z_1}\frac{P_1(z_1, s_2)\, dz_1}{(s_1 - z_1)\, M_2(z_1, s_2)} \tag{6.3.1}$$

for $\sigma_1 > \mathrm{Re}(z_1) > -\mathrm{Re}(a_1)$, with Z_1 as in (2.9), and that

$$P_1(s) = (2\pi i)^{-1}\cdot \int\limits_{Z_1}\int\limits_{Z_2}\frac{G_1(z)\, M_2(z)\, M_3(z_1, -a_2)\, dz_1\, dz_2}{(s_1 - z_1)\,(s_2 - z_2)\, M_2(-a_1, z_2)\, M_3(-a_1, -a_2)\, M_3(z)\, M_4(z)} \tag{6.4.1}$$

for $\sigma_j > \mathrm{Re}(z_j) > -a_j$, with Z_1, Z_2 as in (2.8). $G_1(s)$ is given by (4.11.1), and the functions $M_n(s)$ by (5.16).

Next, we assert that the functions $G_2(s)$, $G_4(s)$ are given by

$$G_2(s) = G_1(s)\left[\frac{M_2(s)\, M_3(s_1, -a_2)}{M_2(-a_1, s_2)\, M_3(-a_1, -a_2)} - 1\right], \tag{6.5}$$

$$G_4(s) = G_1(s)\left[\frac{M_3(-a_1, s_2)\, M_4(s)}{M_3(-a_1, -a_2)\, M_4(s_1, -a_2)} - 1\right] \tag{6.6}$$

and that

$$G_3(s) = F(s)\, K(s) - G_1(s) - G_2(s) - G_4(s) \tag{6.7}$$

with F, G_2, G_4 given by (6.3), (6.5), (6.6) respectively.

To verify our assertions (6.3), (6.5), (6.6), (6.7), we must show that one of the four (independently of the other three) implies (6.1), (6.2); and we must also show that each of the four assertions implies the other three. It is however clear that (6.7) is correct if (6.3), (6.5), (6.6) are, so that it will be enough to verify the results for F, G_2, G_4.

Our verification argument will be organized as follows: Section 6.1 gives the verification for $G_2(s)$, omitting a certain proof in order to maintain the continuity of the essential argument; Section 6.2 contains the omitted proof; the results for $F(s)$, $G_4(s)$ are verified in Sections 6.3, 6.4. The argument for $F(s)$ will not only show that (6.3) is equivalent to (6.5) but also that (6.3), independently of (6.5), implies both (6.1) and (6.2). This is more than is logically required, and we therefore observe that the additional material is included for the insight it gives into the structure of our solution.

6.1. Now the result of substituting G_2, as given by (6.5), into the transform equation (4.12) is

$$F(s)\,K(s) = \frac{G_1(s)\,M_2(s)\,M_3(s_1, -a_2)}{M_2(-a_1, s_2)\,M_3(-a_1, -a_2)} + G_3(s) + G_4(s). \tag{6.8}$$

From (6.8) we find, successively,

$$[F\,K]_{\sigma_2^+} = \frac{G_1(s)\,M_2(s)\,M_3(s_1, -a_2)}{M_2(-a_1, s_2)\,M_3(-a_1, -a_2)}, \tag{6.9}$$

$$[F\,K]_1 = [(F\,K)_{\sigma_2^+}]_{\sigma_1^+} = G_1(s). \tag{6.10}$$

The same result is obtained by taking components in the other order: we have

$$[F\,K]_{\sigma_1^+} = G_1(s) + G_4(s), \tag{6.11}$$

$$[F\,K]_1 = [(F\,K)_{\sigma_1^+}]_{\sigma_2^+} = G_1(s), \tag{6.12}$$

so that (6.5) is seen to imply (6.1).

Next, we show that (6.5) implies (6.2). Dividing both sides of (6.8) by $M_3\,M_4$, we have from (5.1) (with the $M_n(s)$ given by (5.16))

$$\left(\frac{i}{2}\right) F\,M_1\,M_2 = \frac{G_1\,M_2\,M_3(s_1, -a_2)}{M_2(-a_1, s_2)\,M_3(-a_1, -a_2)\,M_3\,M_4} + \frac{G_3 + G_4}{M_3\,M_4}. \tag{6.13}$$

We then have additive decompositions of the following forms (A_1, A_2, C_3, C_4 are unspecified analytic functions, and the B_n are known analytic functions; subscripts denote analyticity domains):

$$(i/2)\,F\,M_1\,M_2 = A_1 + A_2, \tag{6.14}$$

$$\frac{G_1\,M_2\,M_3(s_1, -a_2)}{M_2(-a_1, s_2)\,M_3(-a_1, -a_2)\,M_3\,M_4} = B_1 + B_2 + B_3 + B_4, \tag{6.15}$$

$$(G_3 + G_4)\,(M_3\,M_4)^{-1} = C_3 + C_4. \tag{6.16}$$

The decompositions (6.14), (6.15), (6.16) are unique (this is proved in Section 6.2). From the uniqueness, we have

$$A_1 + A_2 = B_1 + B_2, \tag{6.17}$$

$$C_3 + C_4 = -B_3 - B_4 \tag{6.18}$$

where, since the B_n are known, we may solve for F from (6.14), (6.17) and may solve for G_3, G_4 from (6.16), (6.18). We have

$$(i/2)\,F\,M_1\,M_2 = B_1 + B_2, \tag{6.19}$$

which gives (using uniqueness again, which in the present case amounts to carrying out a one-variable WIENER-HOPF argument on the variable s_1)

$$F(s) = \frac{-2i}{M_1(s)} \left[\frac{B_1(s)}{M_2(s)}\right]_{\sigma_1^+} \tag{6.20}$$

with $B_1(s)$ given, according to (6.15), by

$$B_1(s) = \left[\frac{G_1(s)\,M_2(s)\,M_3(s_1, -a_2)}{M_2(-a_1, s_2)\,M_3(-a_1, -a_2)\,M_3(s)\,M_4(s)}\right]_1. \tag{6.21}$$

Our statement that (6.3) implies (6.2) now follows from (6.20) by inspection, since $M_1(s)$ is analytic and nonvanishing in $T(B, 1)$, while $B[{}_1/M_2]_{\sigma_1^+}$ is analytic in $T[(\gamma, 1) \frown (B, 1)]$. Comparing (6.21) with (6.4), we see that $B_1 \equiv P_1$, *i.e.*, our result (6.5) for G_2 implies the result (6.3) for $F(s)$. Notice, again, that the result for $F(s)$ is explicit: see (6.3.1), (6.4.1).

Finally, we use symmetry considerations to show that we must have

$$G_4(s_1, s_2; a_1, a_2) = G_2(s_2, s_1; a_2, a_1), \tag{6.22}$$

and, since it follows from the definitions of the $M_n(s)$ (see (5.16)) that

$$M_2(s_2, s_1) = M_4(s_1, s_2), \tag{6.23}$$

$$M_3(s_2, -a_1) = M_3(-a_1, s_2), \tag{6.24}$$

$$M_2(-a_2, s_1) = M_4(s_1, -a_2), \tag{6.25}$$

we conclude that (6.5) implies (6.6).

6.2. We now complete the argument of Section 6.1 by proving that the decompositions (6.14), (6.15), (6.16) are unique. A proof that $F(s)$, as given by (6.3), is of bounded L_2 norm for $\sigma \in (\gamma, 1)$ will be included.

Consider the left side of (6.15): $G_1(s)$ is given by (4.11.1), and the growths of the $M_n(s)$ are given by (5.23). Denoting the left side of (6.15) by $P(s)$, we then have

$$P(s) = O\left(|s|^{-\frac{7}{4}}\right) \tag{6.26}$$

with

$$|s| = |s_1^2 + s_2^2|^{\frac{1}{2}} \tag{6.26.1}$$

for large $|s|$ in the analyticity domain of $P(s)$, *i.e.* in the tube $T[B \frown (\gamma, 1)]$. $P(s)$ is therefore of bounded L_2 norm for $\sigma \in [B \frown (\gamma, 1)]$ and has according to BOCHNER's theorem (see Section 2) a unique decomposition.

As to (6.14), (6.16), it is sufficient to show that one of these decompositions is unique. Consider the function on the left side of (6.14). We had actually to assume the boundedness of this function's L_2 norm in order to assert the uniqueness of the decomposition (6.14). Since this process led to (6.20), we are logically obliged to show that

$$\|F M_1 M_2\|_2 < \text{const.} \tag{6.27}$$

for $\sigma \in [(\gamma, 1) \frown (B, 1) \frown (B, 2)]$, with $F(s)$ given by (6.20) (or, equivalently, by (6.3)).

To prove that (6.27) holds, we remark first that (6.3) may be shown (and is shown, in Section 6.3; see Eqs. (6.35), (6.36)) to imply

$$F M_1 M_2 = -2i \left[\frac{G_1(s)\, M_2(s)\, M_3(s_1, -a_2)}{M_2(-a_1, s_2)\, M_3(-a_1, -a_2)\, M_3(s)\, M_4(s)}\right]_{\sigma_2^+}. \tag{6.28}$$

From (6.28), we find

$$F M_1 M_2 = O\left(|s|^{-\frac{7}{4}}\right) \tag{6.29}$$

for large $|s|$ in $T[(\gamma, 1) \frown (B, 1) \frown (B, 2)]$, and we see that (6.27) is satisfied. Also from (6.29) and (5.23)

$$F(s) = O\left(|s|^{-\frac{5}{4}}\right), \tag{6.30}$$

so that the function $F(s)$ given by (6.3) is, as required, of bounded L_2 norm for $\sigma \in (\gamma, 1)$.

6.3. We have shown in Sections 6.1, 6.2 that $F(s)$, as given by (6.3), is analytic in $T(\gamma, 1)$ and of bounded L_2 norm for $\sigma \in (\gamma, 1)$. This means that (6.3) implies (6.1). In this section, we prove that (6.3) implies (6.2) and also that (6.3) implies (6.5). The proof that (6.3) implies (6.6) is given in Section 6.4.

Let us begin by writing (in accordance with the notation of (2.6), (2.9))

$$[F M_1 M_2]_{\sigma_1^+} = \frac{1}{2\pi i} \int_{Z_1} \frac{F(z_1, s_2)\, M_1(z_1, s_2)\, M_2(z_1, s_2)\, dz_1}{s_1 - z_1} \tag{6.31}$$

for $\sigma_1 > \operatorname{Re}(z_1) > -\operatorname{Re}(a_1)$; Z_1 denotes a vertical contour from $-i\infty$ to $+i\infty$. Making use of (6.3), (6.4), we rewrite (6.31) as

$$[F M_1 M_2]_{\sigma_1^+} = \frac{-2i}{(2\pi i)^2} \int_{Z_1} \int_{W_1} \frac{M_2(z_1, s_2)\, P_1(w_1, s_2)\, dw_1\, dz_1}{(s_1 - z_1)(z_1 - w_1)\, M_2(w_1, s_2)} \tag{6.32}$$

with $\sigma_1 > \operatorname{Re}(z_1) > \operatorname{Re}(w_1) > -\operatorname{Re}(a_1)$; W_1 denotes a vertical contour from $-i\infty$ to $+i\infty$. Carrying out the integration with respect to z_1, we have, upon deforming the contour to the left,

$$[F M_1 M_2]_{\sigma_1^+} = \frac{-2i}{2\pi i} \int_{W_1} \frac{P_1(w_1, s_2)\, dw_1}{s_1 - w_1} \tag{6.33}$$

for $\sigma_1 > \operatorname{Re}(w_1) > -\operatorname{Re}(a_1)$. A second integration gives

$$[F M_1 M_2]_{\sigma_1^+} = -2i\, P_1(s_1, s_2). \tag{6.34}$$

But (6.34) implies that we have

$$F M_1 M_2 = -2i\,[P_1(s) + P_2(s)] \tag{6.35}$$

with $P_2(s)$ an analytic function in $T(B, 2)$, and we see from (6.4) that

$$P_1 + P_2 = \left[\frac{G_1(s)\, M_2(s)\, M_3(s_1, -a_2)}{M_2(-a_1, s_2)\, M_3(-a_1, -a_2)\, M_3(s)\, M_4(s)}\right]_{\sigma_2^+}. \tag{6.36}$$

Again, (6.36) implies that we have

$$\frac{G_1(s)\, M_2(s)\, M_3(s_1, -a_2)}{M_2(-a_1, s_2)\, M_3(-a_1, -a_2)\, M_3(s)\, M_4(s)} = \sum_{n=1}^{4} P_n(s) \tag{6.37}$$

where P_1, P_2 are as before, and P_3, P_4 are functions analytic in $T(B, 3)$, $T(B, 4)$ respectively.

Now multiply both sides of (6.35) by $(i/2)\, M_3(s)\, M_4(s)$: the result, upon making use of (5.1) on the left side and of (6.37) on the right side, is

$$F K = \frac{G_1 M_2 M_3(s_1, -a_2)}{M_2(-a_1, s_2)\, M_3(-a_1, -a_2)} - (P_3 + P_4)\, M_3 M_4. \tag{6.38}$$

To prove that (6.38) implies (6.2), we proceed exactly as we did in showing that (6.8) implies (6.2). Since (6.3) has been shown to imply (6.38), our proof that (6.3) implies (6.2) is complete.

Notice next that the right side of (6.38) is equal to $G_1+G_2+G_3+G_4$. It is then clear that we have

$$\frac{G_1 M_2 M_3(s_1, -a_2)}{M_2(-a_1, s_2) M_3(-a_1, -a_2)} = G_1 + G_2, \tag{6.39}$$

$$-(P_3+P_4) M_3 M_4 = G_3 + G_4 \tag{6.40}$$

with (6.39) equivalent to (6.5), so that we have proved that (6.3) implies (6.5).

6.4. In this section we prove that (6.3) and (6.6) are equivalent. Since we have demonstrated the equivalence of (6.3) and (6.5) in Sections 6.1, 6.2, 6.3, and have also shown in those sections that the results (6.3), (6.5) separately imply (6.1), (6.2), the arguments of this section will complete our verification.

We begin by noticing that the symmetry of our problem gives

$$F(s_1, s_2; a_1, a_2) = F(s_2, s_1; a_2, a_1). \tag{6.41}$$

Taking account of (6.41) and the symmetry properties of the $M_n(s)$ (see (6.23), (6.24), (6.25)), we find that an alternate (and completely equivalent) form of our result (6.3) is

$$F(s) = \frac{-2i}{M_1(s)} \left[\frac{R_1(s)}{M_4(s)}\right]_{\sigma_8^+} \tag{6.42}$$

with

$$R_1(s) = \left[\frac{G_1(s) M_3(-a_1, s_2) M_4(s)}{M_2(s) M_3(-a_1, -a_2) M_3(s) M_4(s_1, -a_2)}\right]_1. \tag{6.43}$$

The notation in the last two equations is that of (2.7), (2.5), which means (according to (2.10), (2.8)) that (6.42), (6.43) are explicit integral representations.

An argument of precisely the type we used to show that (6.3) implies (6.38) will now show that (6.42) implies

$$F K = \frac{G_1(s) M_3(-a_1, s_2) M_4(s)}{M_3(-a_1, -a_2) M_4(s_1, -a_2)} - (Q_2 + Q_3) M_2 M_3 \tag{6.44}$$

where Q_2, Q_3 are analytic functions in $T(B, 2)$, $T(B, 3)$. We observe that (6.44) implies (6.2).

Next, we have as consequences of (6.44) the equations (analogous to (6.39), (6.40))

$$\frac{G_1(s) M_3(-a_1, s_2) M_4(s)}{M_3(-a_1, -a_2) M_4(s_1, -a_2)} = G_1 + G_4 \tag{6.45}$$

$$-(Q_2 + Q_3) M_2 M_3 = G_2 + G_3. \tag{6.46}$$

Since (6.45) is equivalent to (6.6), we have shown that (6.42) implies (6.6); also, since (6.3) and (6.42) are equivalent, we conclude that (6.3) implies (6.6).

To show that (6.6) implies (6.42) (and therefore that it implies (6.3)), we make use of (6.6) in our transform equation (4.12). The result, after division of both sides by $M_2 M_3$, is

$$\left(\frac{i}{2}\right) F M_1 M_4 = \frac{G_1 M_3(-a_1, s_2) M_4}{M_2 M_3 M_3(-a_1, -a_2) M_4(s_1, -a_2)} + \frac{G_2+G_3}{M_2 M_3}. \tag{6.47}$$

An argument which follows the same course as that which led from (6.13) to (6.20) then gives us

$$F(s) = \frac{-2i}{M_1(s)} \left[\frac{D_1(s)}{M_4(s)}\right]_{\sigma_2^+} \tag{6.48}$$

with $D_1(s) \equiv R_1(s)$ (see (6.43)). The right side of (6.48) is therefore precisely the right side of (6.42), and (6.42) in turn is equivalent to (6.3). Our proof that (6.3) and (6.6) imply one another is complete.

7. Solution of the boundary problem

The results of Section 6 yield a solution of the mixed boundary problem on the quarter-plane formulated in Section 3. We have the following

Theorem. *The expression* (4.1), *with* $K(s)$ *given by* (4.2), *represents a solution to the boundary problem stated in Eqs.* (3.3) *through* (3.9), *provided that* $F(s)$ *is given by* (6.3) (*or, equivalently, by* (6.42)).

Proof. We have shown in Section 6 that $F(s)$ is analytic in $T(\gamma, 1)$ (see (4.6) for $(\gamma, 1)$) and that it is (according to (6.30)) of bounded L_2 norm for $\sigma \in (\gamma, 1)$; also, we observed in Section 4 that $K(s)$ is analytic in $T(B)$ (see (4.7) for B) and uniformly bounded for $\sigma \in B$. Since $[2K(s)]^{-1}$ has negative real part (see the remark following (4.2)), we conclude that the integral on the right side of (4.1) converges uniformly in any region interior to $T[B \cap (\gamma, 1)]$, as do the integrals representing the second derivatives of u. It follows that (4.1) represents an actual solution of (3.3), if $F(s)$ is given by (6.3) (or by (6.42)).

Since (4.1) represents $u(x_1, x_2, x_3)$, we have

$$u(x_1, x_2, 0) = \frac{1}{(2\pi i)^2} \int_{S_1} \int_{S_2} F(s)\, K(s) \exp(s \cdot x)\, d s_1\, d s_2 \tag{7.1}$$

by dominated convergence. For $x_j \geqq 0$,

$$u(x_1, x_2, 0) = \frac{1}{(2\pi i)^2} \int_{S_1} \int_{S_2} [F(s)\, K(s)]_1 \exp(s \cdot x)\, d s_1\, d s_2 \tag{7.2}$$

since

$$\int_{S_1} \int_{S_2} [F(s)\, K(s)]_n \exp(s \cdot x)\, d s_1\, d s_2 = 0 \tag{7.2.1}$$

for $n = 2, 3, 4$ and $x_j \geqq 0$; the notation $[FK]_n$ is defined by (2.5). We proved in Sec. 6.3 that our result for $F(s)$ (*i.e.*, either (6.3) or its equivalent (6.42)) implies (6.1), and we may therefore use (6.1) together with the explicit form (4.11.1) of $G_1(s)$ to rewrite (7.2) as

$$u(x_1, x_2, 0) = \frac{-1}{(2\pi i)^2} \int_{S_1} \int_{S_2} \frac{\exp(s \cdot x)\, d s_1\, d s_2}{(s_1 + a_1)(s_2 + a_2)}. \tag{7.3}$$

Since the right side of (7.3) is equal to $-\exp(-a_1 x_1 - a_2 x_2)$ for $x_j \geqq 0$, we have shown that (3.4) is satisfied.

To verify (3.5), we remark as we did above that $F(s)$ is, from (6.30), of bounded L_2 norm for $\sigma \in (\gamma, 1)$. Plancherel's theorem (see [3], Prop. 4.1) therefore shows that $F(s)$ corresponds, under Laplace inversion, to a function $f(x)$ such that

$$\|f(x) \exp(-s \cdot x)\|_2 = \|F(s)\|_2 \tag{7.4}$$

for $\sigma \in (\gamma, 1)$. The function $f(x)$ is given (with equality in the mean-square sense) by

$$f(x) = \mathcal{L}^{-1}[F(s)] = \frac{1}{(2\pi i)^2} \int\limits_{S_1} \int\limits_{S_2} F(s) \exp(s \cdot x)\, d s_1\, d s_2, \tag{7.5}$$

so that (4.4) is rigorously established (*i.e.*, it is not merely a formal representation), provided that $F(s)$ is given by (6.3) or by (6.42). Also, our function $F(s)$ has been shown to be analytic in $T(\gamma, 1)$, so that the inverse Laplace transform on the right side of (7.5) vanishes for $E[x_1 < 0 \cup x_2 < 0]$, and (3.5) holds.

Since $f(x)$ has been shown to exist, we may verify (3.6) by showing that

$$\lim_{\substack{s_1 \to -a_1 \\ s_2 \to a_2}} (s_1 + a_1)(s_2 + a_2)\, F(s) = \text{const.} \tag{7.6}$$

or, equivalently, that

$$\lim_{\substack{s_1 \to -a_1 \\ s_2 \to -a_2}} (s_1 + a_1)(s_2 + a_2)\, F(s)\, K(s) = \text{const.} \tag{7.7}$$

It is understood that the results (7.6), (7.7) must be independent of the order in which the limits are taken.

Notice first that

$$\lim_{s_1 \to -a_1} (s_1 + a_1)\, F(s)\, K(s) = \lim_{s_1 \to -a_1} (s_1 + a_1)\, [G_1(s) + G_4(s)] \tag{7.8}$$

since G_2, G_3 are analytic at $s_1 = -a_1$, so that

$$\lim_{s_1 \to -a_1} (s_1 + a_1)\, [G_2(s) + G_3(s)] = 0. \tag{7.8.1}$$

Making use of (6.6) and (4.11.1) in (7.8), we have

$$\lim_{s_1 \to -a_1} F(s)\, K(s) = \frac{-1}{(s_2 + a_2)}\, \frac{M_3(-a_1, s_2)\, M_4(-a_1, s_2)}{M_3(-a_1, -a_2)\, M_4(-a_1, -a_2)}, \tag{7.9}$$

and it follows that

$$\lim_{\substack{s_1 \to -a_1 \\ s_2 \to a_2}} (s_1 + a_1)(s_2 + a_2)\, F(s)\, K(s) = -1. \tag{7.10}$$

If we take limits in the other order, we have

$$\lim_{s_2 \to -a_2} (s_2 + a_2)\, F(s)\, K(s) = \lim_{s_2 \to -a_2} (s_2 + a_2)\, [G_1(s) + G_2(s)] \tag{7.11}$$

and therefore find, using (6.5) and (4.11.1), that

$$\lim_{\substack{s_2 \to -a_2 \\ s_1 \to -a_1}} (s_1 + a_1)(s_2 + a_2)\, F(s)\, K(s) = -1. \tag{7.12}$$

The results (7.10), (7.12) are together equivalent to (7.7), as well as to (7.6), and we conclude that (3.6) is satisfied. We remark that (3.7), (3.7.1) may be deduced from (3.6); these conditions may also be verified directly by an argument similar to that we have just given. An alternate verification of (3.7), (3.7.1) will be found in Section 8.

Finally, we make use of (6.30) to prove that $f(x)$ is of integrable square at the origin (*i.e.*, (3.9) holds), and it is true *a fortiori* that (3.8) holds. In Section 8, we give a more detailed account of the behavior of the functions $f(x)$, $u(x_1, x_2, 0)$ near the origin.

8. Properties of the diffracted field

In this section we discuss the behavior of the diffracted field at infinity, at the edges and at the corner of our quarter-infinite screen.

The total field (incident field plus diffracted field) in our problem is

$$u_{\mathrm{tot}}(x_1, x_2, x_3) = u_0(x_1, x_2, x_3) + u(x_1, x_2, x_3) \tag{8.1}$$

with u_0, u given by (3.1), (4.1) respectively. We proved in Section 7 that (4.1) represents a solution to the boundary problem stated in Eqs. (3.3) through (3.9), provided that $F(s)$ is given by (6.3) or, equivalently, by (6.42). Notice now that (4.1) gives (by an application of PARSEVAL's theorem: see (5.9.3))

$$u(x_1, x_2, x_3) = \frac{-1}{4\pi} \int_0^\infty \int_0^\infty \frac{f(x_1^0, x_2^0) \exp(-i k |R - R^0|) \, dx_1^0 \, dx_2^0}{|R - R^0|} \tag{8.2}$$

where $f(x)$ is given by (7.5), and

$$|R - R^0| = +[(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3)^2]^{\frac{1}{2}}.$$

Physically, the statement made by (8.2) is the evident one that the diffracted field results from a distribution of point sources, with density $f(x)$, over the quarter-plane $x_1 \geqq 0$, $x_2 \geqq 0$, $x_3 = 0$. It follows from the representation (8.2) that the diffracted field is outgoing at infinity; this, as we remarked in the third paragraph of Section 3, is equivalent to Sommerfeld's radiation condition. It is also clear from (8.2) that our solution meets conditions (3.7), (3.7.1).

Next, we consider the diffracted field along the edges, but at a great distance from the corner, of the quarter-infinite screen. We shall show that our result (8.2) for the diffracted field reduces, for large positive x_1 (x_2), to the field scattered by Σ_1 (Σ_2) (see (3.10), (3.11) for the semi-infinite screens Σ_1, Σ_2).

The diffracted fields u_{I}, u_{II} corresponding to scattering by Σ_1, Σ_2 are given by expressions analogous to (8.2):

$$u_{\mathrm{I}}(x_1, x_2, x_3) = \frac{-1}{4\pi} \int_0^\infty \int_{-\infty}^\infty \frac{\varphi_{\mathrm{I}}(x_1^0, x_2^0) \exp(-i k |R - R^0|) \, dx_1^0 \, dx_2^0}{|R - R^0|}, \tag{8.3}$$

$$u_{\mathrm{II}}(x_1, x_2, x_3) = \frac{-1}{4\pi} \int_{-\infty}^\infty \int_0^\infty \frac{\varphi_{\mathrm{II}}(x_1^0, x_2^0) \exp(-i k |R - R^0|) \, dx_1^0 \, dx_2^0}{|R - R^0|}, \tag{8.4}$$

where $\varphi_{\mathrm{I}}(x)$, $\varphi_{\mathrm{II}}(x)$ are current densities on Σ_1, Σ_2; it is understood that the screens Σ_1, Σ_2 are perfectly conducting and that the incident field is the plane wave u_0 of (3.1). Applying the appropriate boundary condition (that the total field vanish on Σ_1, Σ_2), we have, as equations from which φ_{I}, φ_{II} may be deter-

mined,

$$\exp(-a\cdot x)=\frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty\frac{\varphi_{\mathrm{I}}(x_1^0,x_2^0)\exp(-ik|r-r^0|)\,dx_1^0\,dx_2^0}{|r-r^0|},\qquad(8.5.1)$$

$$\exp(-a\cdot x)=\frac{1}{4\pi}\int_{-\infty}^\infty\int_0^\infty\frac{\varphi_{\mathrm{II}}(x_1^0,x_2^0)\exp(-ik|r-r^0|)\,dx_1^0\,dx_2^0}{|r-r^0|},\qquad(8.5.2)$$

where $a\cdot x=a_1x_1+a_2x_2$, $|r-r^0|=+[(x_1-x_1^0)^2+(x_2-x_2^0)^2]^{\frac{1}{2}}$.

We shall now calculate the Laplace transforms $\Phi_{\mathrm{I}}(s)$, $\Phi_{\mathrm{II}}(s)$ of the functions $\varphi_{\mathrm{I}}(x)$, $\varphi_{\mathrm{II}}(x)$ which satisfy (8.5), (8.6) and shall show that $F(s)$, as given by (6.3), reduces to $\Phi_{\mathrm{I}}(s)$, $\Phi_{\mathrm{II}}(s)$ in the appropriate limiting cases. To begin with it is clear from physical considerations that the solution $\varphi_{\mathrm{I}}(x)$ of (8.5.1) has the form

$$\varphi_{\mathrm{I}}(x_1,x_2)=\exp(-a_1x_1)\,\psi_{\mathrm{I}}(x_2)\qquad(8.6.1)$$

and also that the solution $\varphi_{\mathrm{II}}(x)$ of (8.5.2) has the form

$$\varphi_{\mathrm{II}}(x_1,x_2)=\exp(-a_2x_2)\,\psi_{\mathrm{II}}(x_1).\qquad(8.6.2)$$

The function $\psi_{\mathrm{I}}(x_2)$ is then to be determined from the equation

$$\exp(-a_2x_2)=(i/4)\int_0^\infty\psi_{\mathrm{I}}(x_2^0)\,H_0^{(2)}(k_1|x_2-x_2^0|)\,dx_2^0\qquad(8.7.1)$$

where $k_1=(k^2+a_1^2)^{\frac{1}{2}}$; and $\psi_{\mathrm{II}}(x_1)$ is to be determined from

$$\exp(-a_1x_1)=(i/4)\int_0^\infty\psi_{\mathrm{II}}(x_1^0)\,H_0^{(2)}(k_2|x_1-x_1^0|)\,dx_1^0\qquad(8.7.2)$$

where $k_2=(k^2+a_2^2)^{\frac{1}{2}}$.

The Laplace transforms

$$\Psi_{\mathrm{I}}(s_2)=\int_0^\infty\psi_{\mathrm{I}}(x_2)\exp(-s_2x_2)\,dx_2,\qquad(8.8.1)$$

$$\Psi_{\mathrm{II}}(s_1)=\int_0^\infty\psi_{\mathrm{II}}(x_1)\exp(-s_1x_1)\,dx_1\qquad(8.8.2)$$

of the solutions of (8.7.1), (8.7.2) are given by

$$\Psi_{\mathrm{I}}(s_2)=2i(s_2+a_2)^{-1}(-a_2-ik_1)^{\frac{1}{2}}(s_2+ik_1)^{\frac{1}{2}},\qquad(8.9.1)$$

$$\Psi_{\mathrm{II}}(s_1)=2i(s_1+a_1)^{-1}(-a_1-ik_2)^{\frac{1}{2}}(s_1+ik_2)^{\frac{1}{2}},\qquad(8.9.2)$$

so that (from (8.6.1), (8.6.2))

$$\Phi_{\mathrm{I}}(s)=(s_1+a_1)^{-1}\Psi_{\mathrm{I}}(s_2),\qquad(8.10.1)$$

$$\Phi_{\mathrm{II}}(s)=(s_2+a_2)^{-1}\Psi_{\mathrm{II}}(s_1).\qquad(8.10.2)$$

The results (8.10.1), (8.10.2) are known to be the transforms of the unique physically acceptable current densities, in the respective cases of diffraction by the perfectly conducting semi-infinite screens Σ_1, Σ_2. Our task is to show that the function $f(x)$ given by (7.5) is indistinguishable from φ_{I} (φ_{II}) for sufficiently large positive x_1 (x_2). This means, according to (8.6.1), (8.6.2), that we must

show that

$$\lim_{x_1\to+\infty} \exp(a_1 x_1) f(x) = \psi_{\mathrm{I}}(x_2), \tag{8.11.1}$$

$$\lim_{x_2\to+\infty} \exp(a_2 x_2) f(x) = \psi_{\mathrm{II}}(x_1). \tag{8.11.2}$$

or, equivalently, that

$$\lim_{s_1\to-a_1} (s_1+a_1) F(s) = \Psi_{\mathrm{I}}(s_2), \tag{8.12.1}$$

$$\lim_{s_2\to-a_2} (s_2+a_2) F(s) = \Psi_{\mathrm{II}}(s_1), \tag{8.12.2}$$

where $F(s)$ is given by (6.3) (or (6.42)) and Ψ_{I}, Ψ_{II} by (8.9.1), (8.9.2).

From (7.9), we have (in the notation of (5.21), (5.22))

$$\lim_{s_1\to-a_1} (s_1+a_1) F(s) = \frac{2i(s_2+a_2)^{-1}}{M^+(-a_1, s_2)\, M^-(-a_1, -a_2)}. \tag{8.13}$$

Since we have (with $k_1 = (k^2+a_1^2)^{\frac{1}{2}}$)

$$M^+(-a_1, s_2) = (s_2 + i k_1)^{-\frac{1}{2}}, \tag{8.14.1}$$

$$M^-(-a_1, -a_2) = (-a_2 - i k_1)^{-\frac{1}{2}}, \tag{8.14.2}$$

we see that (8.13) is equivalent to (8.12.1), with $\Psi_{\mathrm{I}}(s_2)$ given by (8.9.1). A result analogous to (7.9) follows from (6.5):

$$\lim_{s_2\to-a_2} (s_2+a_2) F(s)\, K(s) = \frac{-1}{(s_1+a_1)}\,\frac{M_2(s_1, -a_2)\, M_3(s_1, -a_2)}{M_2(-a_1, -a_2)\, M_3(-a_1, -a_2)}, \tag{8.15}$$

and we see from this that

$$\lim_{s_2\to-a_2} (s_2+a_2) F(s) = \frac{2i(s_1+a_1)^{-1}}{M_+(s_1, -a_2)\, M_-(-a_1, -a_2)} \tag{8.16}$$

with M_+, M_- defined by (4.14), (4.15), and related to the M_n by (5.17), (5.18). We also have (with $k_2 = (k^2+a_2^2)^{\frac{1}{2}}$)

$$M_+(s_1, -a_2) = (s_1 + i k_2)^{-\frac{1}{2}}, \tag{8.17.1}$$

$$M_-(-a_1, -a_2) = (-a_1 - i k_2)^{-\frac{1}{2}}, \tag{8.17.2}$$

so that (8.16) is equivalent to (8.12.2) with $\Psi_{\mathrm{II}}(s_1)$ given by (8.9.2).

The argument we have given shows that the diffracted field u of (8.2) reduces, for large positive x_1 (x_2), to the diffracted field u_{I} of (8.2) (u_{II} of (8.3)) corresponding to the incidence of u_0 on Σ_1 (Σ_2). Also, the diffracted fields u_{I}, u_{II} are the unique physically acceptable fields associated with the diffraction by Σ_1, Σ_2 respectively. This means that u_{I}, u_{II} have the known energy-conserving behavior near the respective edges $x_2=0$, $x_1=0$ (*i.e.*, $u_{\mathrm{I}} \sim r^{\frac{1}{2}}$ near $x_2=0$, $u_{\mathrm{II}} \sim r^{\frac{1}{2}}$ near $x_1=0$). The behavior of our diffracted field u (as given by (8.2), with $f(x)$ given by (7.5)) near the edges of the quarter-infinite screen may accordingly be described as follows: $u \sim r^{\frac{1}{2}}$ near $x_1=0$, for sufficiently large $x_2>0$; and $u\sim r^{\frac{1}{2}}$ near $x_2=0$, for sufficiently large $x_1>0$. Further, the current density $f(x)$ (as given by (7.5)) has the corresponding $r^{-\frac{1}{2}}$ behavior near either edge, at sufficiently large distances from the corner.

We conclude by discussing the behavior of the diffracted field near the corner (and in the plane) of our quarter-infinite screen. The quantities of interest will be $u(x_1, x_2, 0)$, evaluated as $r=(x_1^2+x_2^2)^{\frac{1}{2}}$ approaches zero along a path exterior to the screen, and $f(x_1, x_2)$, evaluated as r approaches zero along a path interior to the screen.

From (7.1), the Laplace transform of $u(x_1, x_2, 0)$ is FK; we accordingly deduce the behavior of u near the origin from that of $(s_1 s_2 FK)$ near infinity. Results for two special modes of approach follow at once: from (6.5), the result as $r\to 0$ along the negative x_1-axis is seen to be

$$u(x_1, x_2, 0) \sim r^{\frac{1}{4}}, \tag{8.18}$$

and we also see, from (6.6), that the same result is obtained as $r\to 0$ along the negative x_2-axis. The corresponding results for $f(x_1, x_2)$ are clear: we have

$$f(x_1, x_2) \sim r^{-\frac{3}{4}} \tag{8.19}$$

as $r\to 0$ along the positive x_1-axis or along the positive x_2-axis.

An approximation method developed by NOBLE [4] has led to results which differ from (8.18), (8.19): the exponents obtained in [4] are .30, $-.70$ rather than .25, $-.75$. We have, however, been informed [5] that the exponents obtained by the method of [4] vary with the mode of approach to the corner and, in particular, that the results mentioned are valid for an approach along $x_1=x_2$ (rather than along $x_1=0$ or $x_2=0$). A similar variation in field behavior may (or may not) be discernible in the course of an exact analysis. We are presently continuing our investigation in an effort to clarify this matter.

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Fehlerabschätzungs- und Eindeutigkeitssätze für Integro-Differentialgleichungen

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Übersicht

In der folgenden Untersuchung werden die Lösungen $u(t)$ der allgemeinen impliziten Operatoren-Differentialgleichung

$$f(t, u', u, Fu) = 0 \quad \text{in} \quad 0 < t \leqq T \tag{1}$$

unter der Anfangsbedingung

$$u(0) = \eta \tag{2}$$

abgeschätzt und Einzigkeitskriterien für das Anfangswertproblem (1), (2) angegeben. Darin sei Fu ein Operator, der eine auf $0 \leqq t \leqq T$ stetige Funktion u in eine dort definierte Funktion $v = Fu$ transformiert; die üblichen Integraloperatoren besitzen im allgemeinen diese Eigenschaft. Über die Lösbarkeit des Problems (1), (2) werden jedoch keine Aussagen gemacht werden.

Das Anfangswertproblem (1), (2) enthält u.a. die folgenden Sonderfälle:

a) Die gewöhnliche explizite Differentialgleichung erster Ordnung

$$u'(t) = g(t, u) \quad \text{mit} \quad u(0) = \eta, \tag{3}$$

ferner die aus (3) folgende „integrierte Form"

$$u(t) = \eta + \int_0^t g\big(s, u(s)\big)\, ds,$$

sowie mit $z(t) = u'(t)$ die Gestalt

$$z(t) = g\left(t, \eta + \int_0^t z(s)\, ds\right).$$

b) Die Fredholmsche (nichtlineare) Integralgleichung zweiter Art

$$u(t) = \int_0^T K\big(s, t, u(s)\big)\, ds + g(t),$$

in der für spezielle Kerne K auch die Volterrasche Integralgleichung enthalten ist.

c) Die aus der Anfangswertaufgabe für die explizite Differentialgleichung zweiter Ordnung

$$u''(t) = g(t, u, u'), \qquad u(0) = \eta, \qquad u'(0) = \zeta$$

folgenden „integrierten Formen"

$$u'(t) = \zeta + \int_0^t g\big(s, u(s), u'(s)\big)\, ds, \qquad u(0) = \eta$$

und $\big($mit $z(t) = u'(t)\big)$

$$z'(t) = g\Big(t, \eta + \int_0^t z(s)\, ds, z(t)\Big), \qquad z(0) = \zeta.$$

Wie man aus diesen Beispielen sieht, kann die Anfangsbedingung (2) unter Umständen in der Gleichung (1) mitenthalten sein.

Die der nachfolgenden Untersuchung zugrundeliegende Abschätzungsmethode ist im ersten Abschnitt I enthalten. Vereinfacht dargestellt lautet sie: Ist Fu ein „monoton wachsender" Operator und genügt die Funktion f bestimmten Monotoniebedingungen, dann ist die Aufgabe (1), (2) „von monotoner Art" (vgl. L. COLLATZ [*1*]), d. h. es gilt das folgende Lemma:

Aus

$$v(0) < u(0) = \eta < w(0)$$

und

$$f(t, v', v, Fv) < f(t, u', u, Fu) = 0 < f(t, w', w, Fw)$$

folgt stets

$$v(t) < u(t) < w(t) \quad \textit{auf ganz} \quad 0 \leqq t \leqq T.$$

Lassen sich also zwei solche Funktionen v und w mit den genannten Eigenschaften angeben, dann sind v und w untere und obere Schranken für jede Lösung u des Anfangswertproblems (1), (2). Für gewisse Volterrasche Integral- und Integro-Differentialgleichungen sind die Voraussetzungen dieses Abschnitts erfüllt, die Abschätzung ihrer Lösungen wird also durch dieses Lemma unmittelbar geleistet.

Der nächste Abschnitt II befaßt sich mit der Fehlerabschätzung einer Näherungslösung $v(t)$ von (1), (2), für die allein der „Anfangsfehler" $e = v(0) - \eta$ und der „Defekt" $d(t) = f(t, v', v, Fv)$ bekannt ist. Gesucht ist eine Abschätzung für den Fehler $v(t) - u(t)$ allein aus der Kenntnis von e und $d(t)$ („Defektabschätzung"). Im allgemeinen hat der Defekt $d(t)$ nicht durchweg gleiches Vorzeichen und auch Fu und f genügen für gewöhnlich nicht den Monotonievoraussetzungen des Abschnitts I, so daß sich das Lemma nicht direkt anwenden läßt. Gibt es nun zu (1), (2) ein „Vergleichsproblem"

$$\omega(t, \varrho', \varrho, \Omega\varrho) > d(t), \qquad \varrho(0) > e,$$

dessen Operator Ω und dessen Funktion w den Voraussetzungen des Lemmas genügen, und sind ω und Ω in einem gewissen Sinne Majoranten für f und F, dann gilt

$$|v(t) - u(t)| < \varrho(t) \quad \text{in} \quad 0 \leqq t \leqq T,$$

d. h. eine Lösung $\varrho(t)$ dieses Vergleichsproblems ist eine Schranke für den absoluten Fehler der Näherungsfunktion $v(t)$.

Läßt sich nun zeigen, daß die von e und $d(t)$ abhängige Funktion $\varrho(t)$ für $e \to 0$ und $d \to 0$ ebenfalls verschwindet, so heißt das, daß der Abstand $v - u$ zweier Lösungen v und u von (1), (2) beliebig klein wird, d. h. daß das Anfangswertproblem (1), (2) höchstens eine einzige Lösung besitzt. Der letzte Abschnitt III gibt solche Einzigkeitskriterien an.

Die erwähnten Ergebnisse lassen sich sofort auf Operatoren-Differentialgleichungen höherer Ordnung sowie auf Systeme davon übertragen, der Einfachheit halber soll jedoch von dieser Verallgemeinerung abgesehen werden. Eine Übertragung auf Funktionen mehrerer Veränderlicher ist ebenfalls möglich; eine Theorie für hyperbolische Differentialgleichungen — die im wesentlichen auf denselben Grundideen basiert wie die folgenden Betrachtungen — wurde vor kurzem von W. WALTER in dieser Zeitschrift veröffentlicht [7]. Bei der Ausarbeitung der nachfolgenden Ergebnisse war mir der Rat von Herrn Doz. Dr. W. WALTER sehr wertvoll.

I. Ober- und Unterfunktionen bei Operatoren-Differentialgleichungen mit monotonen Operatoren

Bezeichnungen und Definitionen. Es sei $T>0$ eine Konstante. Das abgeschlossene Intervall $0\leqq t\leqq T$ der reellen Veränderlichen t sei mit I, die links offene Strecke $0<t\leqq T$ mit I_0 bezeichnet. Sämtliche im folgenden eingeführten Funktionen seien als reellwertig und eindeutig erklärt vorausgesetzt.

Der Operator F sei auf dem Raum $C(I)$ der auf I stetigen Funktionen $u(t)$ erklärt, genauer ist $u\in C(I)$ und $v=Fu$, so sei $v(t)$ definiert für alle $t\in I_0$. Die Gesamtheit dieser Operatoren F werde mit $\mathfrak{F}$ bezeichnet. Die Teilmenge $\mathfrak{F}_+\subset\mathfrak{F}$ $[\mathfrak{F}_-\subset\mathfrak{F}]$ der (schwach) monoton wachsenden [fallenden] Operatoren sei erklärt durch die Eigenschaft: Sind $u, v\in C(I)$ und ist für eine beliebige Stelle $s\in I_0$ stets

$$u(t)<v(t) \quad \text{in} \quad 0<t<s, \tag{4}$$

dann soll sogar

$$Fu\leqq[\geqq]\,Fv \quad \text{für} \quad t=s \tag{5}$$

gelten. Ist $F\in\mathfrak{F}_+$, so ist $-F\in\mathfrak{F}_-$. Für die nachfolgenden Untersuchungen genügt es daher, allein den Fall $F\in\mathfrak{F}_+$ zu betrachten.

Beispiel 1. Die folgenden Operatoren gehören zu $\mathfrak{F}_+$, jedes Beispiel enthält eine umfassendere Funktionenklasse. Die verwendeten Integrale sind im Sinne einer der gebräuchlichen (auch uneigentlichen) Integraldefinitionen aufzufassen.

a) $$Fu\equiv\int_0^t u(s)\,ds,$$

b) $$Fu\equiv\int_0^t K(s,t)\,u(s)\,ds,$$

wenn $K(s,t)\geqq 0$ ist für $s, t\in I_0$ und wenn $K(s,t)$ für alle $t\in I_0$ integrierbar über s ist.

c) $$Fu\equiv\int_0^t K\big(s,t,u(s)\big)\,ds,$$

wenn $K(s,t,z)-K(s,t,\bar z)\geqq 0$ bleibt für $s, t\in I_0$ und alle $z\geqq\bar z$ und wenn $K\big(s,t,u(s)\big)$ für $t\in I_0$ und alle Funktionen $u\in C(I)$ integrierbar über s ist.

Definition. Die Menge $Z=Z(I)$ der auf I zulässigen Funktionen sei die Gesamtheit aller auf I stetigen, in I_0 differenzierbaren Funktionen $u(t)$.

Problemstellung. Die Funktion $f(t, x, y, z)$ sei definiert für $t\in I_0$, $-\infty<x, y, z<+\infty$. Man wählt einen Operator $F\in\mathfrak{F}$ und eine reelle Zahl η. Ist die Funktion $u\in Z(I)$ und genügt $u(t)$ der Gleichung

$$f(t, u', u, Fu)=0 \quad \text{in } I_0 \tag{1}$$

und der Anfangsbedingung

$$u(0)=\eta, \tag{2}$$

dann heißt u Lösung des Anfangswertproblems (1), (2).

Definition. Ist die Funktion $f(t, x, y, z)$ definiert für $t\in I_0$, $-\infty< x, y, z<+\infty$ und gilt dort sogar

$$f(t, x, y, z)-f(t, \bar{x}, y, \bar{z})\geqq 0 \quad \text{für} \quad x\geqq\bar{x}, \qquad z\leqq\bar{z}, \tag{6}$$

d.h. ist f dort (schwach) monoton wachsend in x und (schwach) monoton fallend in z, so soll dafür $f\in M$ geschrieben werden.

Satz 1. *Es sei $f\in M$, für die Funktionen $v, w\in Z$ gelte*

$$v(0)<w(0) \tag{7}$$

und mit dem Operator $F\in\mathfrak{F}_+$ sei noch

$$f(t, v', v, Fv)<f(t, w', w, Fw) \quad \textit{in } I_0. \tag{8}$$

Dann ist sogar

$$v(t)<w(t) \quad \textit{auf ganz } I. \tag{9}$$

Beweis durch Antithese. Angenommen, die Behauptung (9) sei falsch, dann gibt es Stellen $t'\in I$ mit $v(t')\geqq w(t')$. Man bildet $\bar{t}=\inf t'$. Wegen der Stetigkeit von v und w gilt dann

$$v(\bar{t})=w(\bar{t}), \tag{10}$$

nach (7) ist also insbesondere $\bar{t}>0$. Wegen $v(t)<w(t)$ in $0\leqq t<\bar{t}$ und wegen $F\in\mathfrak{F}_+$ ist dann

$$Fv\leqq Fw \quad \text{für} \quad t=\bar{t}. \tag{11}$$

Ebenfalls gilt damit

$$\frac{v(\bar{t})-v(t)}{\bar{t}-t}>\frac{w(\bar{t})-w(t)}{\bar{t}-t} \quad \text{für alle } t \text{ aus} \quad 0\leqq t<\bar{t};$$

für $t\to\bar{t}$ folgt daraus wegen der Differenzierbarkeit von v und w auch

$$v'(\bar{t})\geqq w'(\bar{t}). \tag{12}$$

Wegen $f\in M$ findet man damit für die Stelle $t=\bar{t}$ aus (10) bis (12) die Ungleichung $f(\bar{t}, v', v, Fv)\geqq f(\bar{t}, w', w, Fw)$ im Gegensatz zu (8), und aus diesem Widerspruch folgt die Behauptung.

Aus dem Beweis sieht man sofort den

Zusatz. Hängt die Funktion $f(t, x, y, z)$ von x nicht ab, d.h. entartet die Operatoren-Differentialgleichung (1) zu einer reinen Operatorengleichung, so genügt es für die Gültigkeit von Satz 1, wenn die Funktionen $v, w\in C(I)$ sind.

Man benötigt diesen Zusatz z.B. für das folgende

Beispiel 2. (Lemma von GRONWALL, vgl. G. SANSONE [5], S.30) Es seien $K\geqq 0$, $A\geqq 0$, B und $\gamma>0$ vier Konstanten. Die Funktion $v\in C(I)$ genüge auf I der Integralungleichung

$$v(t)-\int_0^t\big(Kv(s)+A\big)\,ds-B\leqq 0. \tag{13}$$

Um Satz 1 anwenden zu können, wird gesetzt:

$$f(t, x, y, z) \equiv y - z - B, \qquad \eta = B, \qquad F u \equiv \int_0^t \bigl(K u(s) + A\bigr)\, ds;$$

es ist $f \in M$ und $F \in \mathfrak{F}_+$. Für die auf I zulässige Funktion

$$w(t) = (A t + B) \exp K t + \gamma \exp 2 K t$$

gilt offenbar

$$f(t, x, w, F w) = \begin{cases} \gamma & \text{für} \quad K = 0 \\ \dfrac{A}{K}(\exp K t - 1 - K t) + \dfrac{\gamma}{2}(1 + \exp 2 K t) & \text{für} \quad K > 0, \end{cases}$$

also $f(t, x, w, F w) > 0$ für $t \in I_0$ und beliebige Werte x; weiter ist $w(0) = B + \gamma > B \geqq u(0)$. Also ist nach Satz 1 stets $u(t) < v(t)$ auf I. Dies gilt für alle Werte $\gamma > 0$, in der Grenze für $\gamma \to 0$ ist also

$$v(t) \leqq (A t + B) \exp K t \quad \text{auf } I$$

für jede Funktion $v \in C(I)$, die der Integralungleichung (13) genügt.

Wird in Satz 1 in der Bedingung (7) das Gleichheitszeichen zugelassen, so ist die Folgerung (9) i.a. nicht mehr richtig, wie schon einfache Beispiele zeigen. Um die Forderung (7) trotzdem abmildern zu können, führt man eine neue Bezeichnung ein gemäß der folgenden

Definition. Sind zwei Funktionen $v(t)$ und $w(t)$ in I_0 erklärt und existiert eine Nullfolge s_ν $(\nu = 1, 2, \ldots)$, für die $s_\nu \in I_0$, $\lim\limits_{\nu > \infty} s_\nu = 0$ und $v(s_\nu) < w(s_\nu)$ gilt für alle $\nu = 1, 2, \ldots$, so soll dafür abkürzend

$$v(0+) < w(0+) \qquad \bigl(\text{bzw. } w(0+) > v(0+)\bigr)$$

geschrieben werden.

Beispiel 3. Sind $v, w \in C(I)$ und ist $v(0) < w(0)$, so gilt offenbar auch $v(0+) < w(0+)$. Sind v, w in einer Umgebung von $t = 0$ differenzierbar und ist $v(0) = w(0)$, aber $v'(0) < w'(0)$, so ist ebenfalls $v(0+) < w(0+)$. Weitere hinreichende Bedingungen lassen sich leicht angeben.

Mit dieser Definition läßt sich die Voraussetzung (7) noch abschwächen, man kann $v(0) = w(0)$ zulassen und allein $v(0+) < w(0+)$ fordern. Da man den Satz 1 auf jedes der Intervalle $s_\nu \leqq t \leqq T$ anwenden kann, gilt (9) noch in ganz I_0. Auch die Bedingung (8) kann noch gemildert werden. Wie man aus dem Beweis sieht, genügt es, sie allein für die dort benutzte Stelle $\bar{t}$ vorauszusetzen. Damit gilt der

Satz 2. *Es sei $f \in M$, für die Funktionen $v, w \in Z$* gelte*

$$v(0+) < w(0+) \tag{7a}$$

und mit dem Operator $F \in \mathfrak{F}_+$ sei noch

$$f(\bar{t}, v', v, F v) < f(\bar{t}, w', w, F w) \tag{8a}$$

* Offenbar würde es mit der Bedingung (7a) genügen, wenn v, w als in I_0 stetig und differenzierbar vorausgesetzt würden. Die Stetigkeit der zulässigen Funktionen an der Stelle $t = 0$ könnte also aufgegeben werden. Da diese Bemerkung für die praktischen Anwendungen nicht sehr wichtig ist, und bei einem Teil der folgenden Sätze auf die Stetigkeit der zulässigen Funktionen an der Stelle $t = 0$ nicht verzichtet werden kann, wurde auf die Mitnahme dieser Verallgemeinerung verzichtet.

für alle Stellen $\bar{t}\in I_0$, für die $v(\bar{t})=w(\bar{t})$ und $v(t)<w(t)$ für $0<t<\bar{t}$ gilt. Dann ist sogar

$$v(t)<w(t) \quad \textit{in ganz } I_0. \tag{9a}$$

Bemerkung. Kennt man die in den Sätzen 1 und 2 auftretenden Funktionen v und w explizit, so ist die Aussage (9) bzw. (9a) i.a. uninteressant, weil man mit der Kenntnis von v und w natürlich auch weiß (oder wenigstens stets prinzipiell nachprüfen kann), ob $v<w$ ist oder nicht. Ihre Bedeutung erhalten die beiden Sätze erst dadurch, daß man sie auf die i.a. noch unbekannten Lösungen des Anfangswertproblems (1), (2)* anwenden kann. In diesem Falle ist — analog zu einer Definition von O. Perron [*3*] bei gewöhnlichen Differentialgleichungen — die folgende Setzung zweckmäßig:

Definition. Eine in I zulässige Funktion $v(t)$ werde als Oberfunktion (bezüglich des Anfangswertproblems (1), (2)) bezeichnet, wenn

$$v(0+)>u(0+) \tag{14}$$

ist für jede Lösung $u(t)$ von (1), (2) und wenn

$$f(t,v',v,Fv)>0 \quad \text{in } I_0 \tag{15}$$

gilt. Sind dagegen (14) und (15) beide mit dem $<$-Zeichen erfüllt, so soll v eine Unterfunktion heißen.

Nach Satz 2 gilt dann das

Lemma 1. *Für jede beliebige Lösung $u(t)$ von* (1), (2), *eine Unterfunktion $v(t)$ und eine Oberfunktion $w(t)$ zu* (1), (2) *gilt stets*

$$v(t)<u(t)<w(t) \quad \textit{in } I_0,$$

falls $f\in M$ und $F\in\mathfrak{F}_+$ ist.

Bemerkung. Dieses Lemma bildet die Grundlage für die folgenden Abschätzungs- und Eindeutigkeitssätze; in der von Herrn Collatz eingeführten Sprechweise ist dann das Anfangswertproblem (1), (2) „von monotoner Art". Zur Abschätzung einer Lösung genügt es danach, passende Ober- und Unterfunktionen anzugeben. Läßt sich sogar je eine Ober- und eine Unterfunktion mit beliebig kleinem Abstand finden, dann hat die Aufgabe (1), (2) höchstens eine einzige Lösung.

Beispiel 4 (Plattengrenzschicht). Zur Untersuchung der Singularität der Blasiusschen Differentialgleichung der Plattengrenzschicht betrachtet man (vgl. B. Punnis [*4*]) das Anfangswertproblem

$$y'''+yy''=0, \qquad y(0)=y'(0)=0, \qquad y''(0)=1 \tag{16}$$

in $0\leqq t<\infty$. Setzt man $u(t)=y'(t)$, dann genügt u offenbar der mit (16) gleichwertigen nichtlinearen Volterraschen Integro-Differentialgleichung

$$f(t,u',u,Fu)\equiv u'-\exp\int_0^t (t-s)\,u(s)\,ds=0 \tag{17}$$

unter der Anfangsbedingung

$$u(0)=0. \tag{18}$$

* Wobei die Gleichung (1) auch durch eine Ungleichung ersetzt werden darf, vgl. Beispiel 2.

Setzt man $Fu \equiv \exp \int_0^t (t-s)\, u(s)\, ds$, so ist $F \in \mathfrak{F}_+$ und $f(t, x, y, z) \equiv x - z$ genügt der Bedingung (6), d.h. es ist $f \in M$. Wie man leicht nachprüft, hat eine Lösung von (17), (18) bei $t=0$ die Entwicklung $u(t) = t + t^4/24 + \cdots$. Eine einfache Oberfunktion ist

$$w(t) = 3(\alpha - t)^{-2} - 3\alpha^{-2} \quad \text{mit} \quad \alpha = \sqrt[3]{6}$$

in dem Intervall $0 \leqq t < \alpha$, denn es ist $w(t) = t + 3t^2/2\alpha + \cdots$, also $u(0+) < w(0+)$, und

$$f(t, w', w, Fw) = 6(\alpha - t)^{-3}[1 - \exp - \alpha t(2\alpha + t)/4] > 0 \quad \text{in} \quad 0 < t < \alpha.$$

Eine Unterfunktion ist

$$v(t) = \begin{cases} t & \text{für} \quad 0 \leqq t \leqq \alpha \\ 3(2\alpha - t)^{-2} + \frac{1}{2}\alpha & \text{für} \quad \alpha \leqq t < 2\alpha \end{cases}$$

wegen $v(0+) < u(0+)$ und

$$f(t, v', v, Fv) = \left\{ \begin{matrix} 1 - \exp \frac{1}{6} t^3 \\ 6(2\alpha - t)^{-3}\left[1 - \exp\left(1 + \frac{1}{4}\alpha(t-\alpha)^2\right)\right] \end{matrix} \right\} < 0 \quad \text{in} \quad 0 < t < 2\alpha.$$

Eine singuläre Stelle jeder Lösung $u(t)$ von (17), (18) liegt damit zwischen $\sqrt[3]{6}$ und $2\sqrt[3]{6}$, bessere Schranken lassen sich leicht angeben.

Bemerkung. Es wäre offenbar nicht nötig gewesen, zur Abschätzung der Lösungen von (17) die vorstehende Theorie zu benutzen, da die Funktion

$$\int_0^t v(s)\, ds \left[\int_0^t w(s)\, ds\right] \qquad \text{Unter- [Ober-] Funktion}$$

zu (17), (18) ist, wenn man die bekannte Theorie der Unter- und Oberfunktionen für Anfangswertaufgaben bei gewöhnlichen Differentialgleichungen heranzieht (vgl. etwa M. Müller [2]). Danach gilt z.B. für jede zweimal differenzierbare Lösung $y(t)$ der Anfangswertaufgabe

$$Dy \equiv y''(t) - g(t, x, y') = 0, \qquad y(0) = \eta, \qquad y'(0) = \zeta \tag{19}$$

stets

$$x(t) < y(t) < z(t) \quad \text{auf } I, \tag{20}$$

wenn $x(t)$ und $z(t)$ auf I zweimal differenzierbar sind, wenn gilt:

$$\begin{gathered} x(0) < \eta < z(0), \qquad x'(0) < \zeta < z'(0), \\ Dx < 0 < Dz \quad \text{in } I_0 \end{gathered} \tag{21}$$

und wenn $g(t, r, s)$ in der Veränderlichen s schwach monoton wächst.

Besitzt g diese Monotonie-Eigenschaft nicht, so ist die Abschätzung (20) i.a. nicht richtig, auch wenn (21) erfüllt ist. Es kann nun sein, daß zwar $g(t, r, s)$ in s *nicht* monoton wächst, daß also eine Lösung $y(t)$ von (19) sich *nicht* durch Unter- und Oberfunktionen abschätzen läßt, daß dagegen $g(t, r, s)$ ein monotones Wachstum bezüglich r zeigt. Dann gestattet aber die mit (19) identische Integro-Differentialgleichung $(u(t) \equiv y'(t))$

$$u'(t) = g\left(t, \eta + \int_0^t u(s)\, ds, u(t)\right), \qquad u(0) = \zeta$$

Abschätzungen durch Unter- und Oberfunktionen $v(t)$ und $w(t)$ nach dem Lemma 1. Damit ist dann natürlich auch

$$y(t) = \eta + \int_0^t u(s)\, ds$$

in Schranken

$$x(t) = \eta + \int_0^t v(s)\, ds \quad \text{und} \quad z(t) = \eta + \int_0^t w(s)\, ds$$

eingeschlossen, obwohl sich diese Abschätzung auf dem direkten Wege nicht zeigen ließ. Wie dieses Beispiel zeigt, kann es also oft für eine Abschätzung zweckmäßig sein, von einer gewöhnlichen Differentialgleichung zu einer Integro-Differentialgleichung überzugehen.

Gegenbeispiel 1. Zu einem Anfangswertproblem (1), (2) braucht es i.a. weder Unterfunktionen noch Oberfunktionen zu geben, selbst wenn $f \in M$ und $F \in \mathfrak{F}_+$ ist. Ist z.B. $f \in M$ und $F \in \mathfrak{F}_+$, hängt $f(t, x, y, z)$ von x nicht ab und ist f zusätzlich noch schwach monoton fallend in y, dann gibt es sicher keine Ober- oder Unterfunktionen zu (1), (2). Denn wäre w eine Oberfunktion, dann gälte nach dem Lemma 1 stets $w > u$ in I_0, also auch $Fw \geqq Fu$ in I_0 wegen $F \in \mathfrak{F}_+$. Wegen der vorausgesetzten Monotonie von f wäre dann auch $0 = f(t, u', u, Fu) \geqq f(t, w', w, Fw)$ in I_0 in Widerspruch zur Bedingung (15) der Oberfunktionen. Entsprechend schließt man für Unterfunktionen. Das Lemma 1 und die daraus zu ziehenden Folgerungen lassen sich also auf diese Klasse von Operatorengleichungen nicht anwenden. Ein Beispiel dafür ist die Klasse der Volterraschen Integralgleichungen erster Art der Gestalt

$$\int_0^t K(s, t, u(s))\, ds = g(t).$$

Es ist für die Anwendungen oft unhandlich, daß in den Ungleichungen (7) und (8) des Satzes 1 (bzw. (7a) und (8a) bei Satz 2) das Gleichheitszeichen nicht zugelassen ist. Man ist dadurch gezwungen, bei der Definition der Unter- und Oberfunktionen in den Bedingungen (14) und (15) das Gleichheitszeichen ebenfalls auszuschließen. Daß diese strengen Forderungen in der Tat notwendig sind, daß also etwa Satz 1 i.a. nicht mehr richtig ist, wenn das $<$-Zeichen in (7) und (8) durch $\leqq$ ersetzt wird — selbst dann nicht, wenn auch in (9) das $\leqq$-Zeichen zugelassen wird — zeigt das folgende

Gegenbeispiel 2. Es sei $T = 2$,

$$f(t, x, y, z) = x - 4\sqrt[3]{y^2} - 4(t-1)^2 z, \qquad Fu = \int_0^t u(s)\, ds;$$

offenbar ist $f \in M$ und $F \in \mathfrak{F}_+$. Man betrachtet also die Integro-Differentialgleichung

$$f(t, u', u, Fu) \equiv u' - 4\sqrt[3]{u^2} - 4(t-1)^2 \int_0^t u(s)\, ds = 0.$$

Für die beiden zulässigen Funktionen $v(t) = (t-1)^3$ und $w(t) \equiv 0$ gilt $v(0) = -1 < 0 = w(0)$ und $f(t, v', v, Fv) = -(t-1)^6 \leqq 0 = f(t, w', w, Fw)$ in $0 \leqq t \leqq 2$, dabei tritt das Gleichheitszeichen allein an der Stelle $t = 1$ auf. Trotzdem ist $v(2) = +1 > 0 = w(2)$, d.h. die Folgerung (9) des Satzes 1 ist falsch, obwohl alle Voraussetzungen von Satz 1 erfüllt sind bis auf die Bedingung (8), die in I sogar nur an der einzigen Stelle $t = 1$ verletzt ist.

Dagegen lassen sich die Bedingungen (7) und (8) von Satz 1 bzw. (7a) und (8a) von Satz 2 noch abschwächen, wenn man die übrigen Voraussetzungen verschärft. Eine besonders einfache Möglichkeit ist die folgende, bei der die schwache Monotonie bei der Definition von $\mathfrak{F}_+$ und M durch eine starke Monotonie ersetzt wird*. Dazu dienen die folgenden

Definitionen. Die Teilmenge $\overline{\mathfrak{F}}_+ \subset \mathfrak{F}_+ \subset \mathfrak{F}$ der stark monoton wachsenden** Operatoren F sei erklärt durch die Eigenschaft: Sind $v, w \in C(I)$ und ist für eine beliebige Stelle $s \in I_0$ stets

$$u(t) < v(t) \quad \text{in} \quad 0 < t < s, \tag{4}$$

dann soll sogar

$$F u < F v \quad \text{für} \quad t = s \tag{5a}$$

gelten. Für die drei Operatoren aus Beispiel 1 gilt offenbar: Zu a): $F \in \overline{\mathfrak{F}}_+$, zu b): $F \in \overline{\mathfrak{F}}_+$, wenn sogar $K > 0$ ist, zu c): $F \in \overline{\mathfrak{F}}_+$, wenn $K(s, t, z) - K(s, t, \bar{z}) > 0$ bleibt für $s, t \in I_0$ und $z > \bar{z}$.

Ist $f(t, x, y, z)$ definiert für $t \in I_0$, $-\infty < x, y, z < +\infty$ und gilt dort

$$f(t, x, y, z) - f(t, \bar{x}, y, \bar{z}) > 0 \quad \text{für} \quad x > \bar{x}, \quad z < \bar{z},$$

d.h. ist f dort stark monoton wachsend in x und stark monoton fallend in z, so soll $f \in \overline{M}$ geschrieben werden.

Damit gilt der folgende Satz 3, in dem gegenüber Satz 2 in den Ungleichungen (8) und (9) noch das Gleichheitszeichen zugelassen wurde, während dafür sogar $f \in \overline{M}$ und $F \in \overline{\mathfrak{F}}_+$ gefordert wird:

Satz 3. *Es sei $f \in \overline{M}$, für die Funktionen $v, w \in Z$ gelte*

$$v(0+) < w(0+) \tag{7a}$$

und mit dem Operator $F \in \overline{\mathfrak{F}}_+$ sei noch

$$f(\bar{t}, v', v, F v) \leqq f(\bar{t}, w', w, F w) \tag{8b}$$

für alle Stellen $\bar{t} \in I_0$, für die $v(\bar{t}) = w(\bar{t})$ und $v(t) < w(t)$ für $0 < t < \bar{t}$ gilt. Dann ist sogar

$$v(t) \leqq w(t) \quad \textit{auf ganz } I. \tag{9b}$$

Der *Beweis* verläuft ganz entsprechend zu dem von Satz 2.

Bei dem Gegenbeispiel 2 war die starke Monotonie von $f(t, x, y, z)$ hinsichtlich z gerade an der „kritischen" Stelle $\bar{t} = 1$ verletzt, sämtliche anderen Voraussetzungen zu Satz 3 waren erfüllt.

Nach Satz 3 liegt es nahe, auch den Begriff der Unterfunktion und der Oberfunktion zu erweitern zu der

Definition. Eine in I zulässige Funktion $v(t)$ werde als Oberfunktion im weiteren Sinne (bezüglich des Anfangswertproblems (1), (2)) bezeichnet, wenn

$$v(0+) < u(0+) \tag{14}$$

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* Es soll jedoch ausdrücklich darauf hingewiesen werden, daß sich auch noch andere hinreichende Bedingungen für diesen Sachverhalt angeben lassen, die im Falle einer reinen Differential- oder einer reinen Integralgleichung sogar ähnlich einfach werden.

** Auf die analog erklärten monoton fallenden Operatoren kann man wie oben für das folgende verzichten.

ist für jede Lösung $u(t)$ von (1), (2) und wenn

$$f(t, v', v, Fv) \geqq 0 \quad \text{in } I_0 \tag{15a}$$

gilt. Sind dagegen (14) und (15a) beide mit dem $<$- bzw. $\leqq$-Zeichen erfüllt, so soll v eine Unterfunktion im weiteren Sinne heißen.

Nach Satz 3 gilt dann das

Lemma 2. *Für jede beliebige Lösung $u(t)$ von (1), (2), eine Unterfunktion im weiteren Sinne $v(t)$ und eine Oberfunktion im weiteren Sinne $w(t)$ zu (1), (2) gilt stets*

$$v(t) \leqq u(t) \leqq w(t) \quad \textit{auf ganz } I,$$

falls $f \in \overline{M}$ und $F \in \overline{\mathfrak{F}}_+$ ist.

II. Fehlerabschätzungen

Ist $v(t)$ eine Näherungslösung zum Anfangswertproblem (1), (2), so ist der „Anfangsfehler" $e := v(0) - \eta \approx 0$ und ebenfalls gilt für den „Defekt" $d(t) := f(t, v', v, Fv) \approx 0$ in I_0. Im folgenden soll allein aus der Kenntnis von Schranken für Anfangsfehler e und Defekt $d(t)$ eine Abschätzung für den Fehler $v(t) - u(t)$ der Näherungslösung $v(t)$ angegeben werden.

Satz 4. *Die Funktion $v \in Z$ habe die Eigenschaften*

$$|v(0) - \eta| \leqq \varepsilon, \tag{22}$$

$$|f(t, v', v, Fv)| \leqq \delta(t) \quad \textit{in } I_0. \tag{23}$$

Die Funktion $\delta(t)$ sei definiert in I_0, $f(t, x, y, z)$ sei erklärt für $t \in I_0$, $-\infty < x, y, z < +\infty$, es sei $F \in \mathfrak{F}$. Weiterhin gelte mit einer Funktion $\omega \in M[\overline{M}]$ und einem Operator $\Omega \in \mathfrak{F}_+[\overline{\mathfrak{F}}_+]$ die Abschätzung

$$f(t, x, w, Fw) - f(t, \bar{x}, \overline{w}, F\overline{w}) \geqq \omega\big(t, x - \bar{x}, w - \overline{w}, \Omega(w - \overline{w})\big) \tag{24}$$

für beliebige Zahlen $x, \bar{x}$ und alle Funktionen $w, \overline{w} \in C(I)$ mit der Eigenschaft $w \geqq \overline{w}$ auf I.

Genügt dann die Funktion $\varrho \in Z$ der Bedingung

$$\omega(t, \varrho', \varrho, \Omega\varrho) > [\geqq]\, \delta(t) \quad \textit{in } I_0 \quad \textit{mit} \quad \varrho(0) > \varepsilon, \tag{25}$$

dann gilt für jede Lösung $u(t)$ des Anfangswertproblems (1), (2) die Abschätzung

$$|v(t) - u(t)| < [\leqq]\, \varrho(t) \quad \textit{auf } I, \tag{26}$$

d.h. $\varrho(t)$ ist eine Schranke für den absoluten Fehler der Näherungslösung $v(t)$ zu (1), (2).

Bemerkung. Während bei den Abschätzungen des vorigen Abschnitts durch Unter- und Oberfunktionen nur Funktionen $f \in M[\overline{M}]$ und Operatoren $F \in \mathfrak{F}_+[\overline{\mathfrak{F}}_+]$ zugelassen werden konnten, brauchen bei der Anwendung des Satzes 4 *keine* solchen Monotonie-Voraussetzungen getroffen zu werden, es genügt vielmehr, wenn allein die „Majoranten" ω und Ω den entsprechenden Bedingungen genügen. Damit ist der Anwendungsbereich von Satz 4 gegenüber den Sätzen von Abschnitt I stark vergrößert worden.

Beweis. Es soll zunächst $v-u<[\leqq]\varrho$ auf I nachgewiesen werden. Man wendet dazu Satz 1 [Satz 3] auf den Ausdruck ω mit dem Operator Ω (statt f mit F) und die beiden Funktionen $v-u$ und ϱ (statt v und w) an. Nach (22) und (25) ist $v(0)-u(0)\leqq\varepsilon<\varrho(0)$, während wegen (23), (24) und (25) stets

$$\omega\big(t, v'-u', v-u, \Omega(v-u)\big) \leqq f(t, v', v, Fv) - f(t, u', u, Fu)$$
$$\leqq \delta(t) < \omega(t, \varrho', \varrho, \Omega\varrho)$$

in I_0 ist. Nach Satz 1 [Satz 3] bleibt dann auch $v(t)-u(t)<[\leqq]\varrho(t)$ auf ganz I, denn ω und Ω genügen den dort verlangten Monotoniebedingungen. Ganz entsprechend beweist man auch $u(t)-v(t)<[\leqq]\varrho(t)$, womit (26) vollständig bewiesen ist.

Beispiel 5 (Lipschitz-Bedingungen). Im folgenden sei die Funktion $L(t)$ in I_0 stetig, das Integral

$$\int_0^t L(s)\,ds \qquad \text{existiere (eventuell uneigentlich)}$$

für $t\in I$.

a) Man betrachtet das Anfangswertproblem der gewöhnlichen Differentialgleichung erster Ordnung

$$u'-g(t,u)=0 \qquad u(0)=\eta, \tag{27}$$

verwendet also etwa $f(t, x, y, z)\equiv x-f(t, y)$, F beliebig. Die Funktion $g(t, y)$ sei dabei definiert für $t\in I_0$, $-\infty<y<+\infty$; die reelle Zahl η sei beliebig. Genügt g der Lipschitz-Bedingung

$$g(t,y)-g(t,\bar{y})\leqq L(t)\,(y-\bar{y}) \quad \text{für} \quad t\in I_0, \qquad y\geqq\bar{y}, \tag{28}$$

dann ist (24) (für einen beliebigen Operator $\Omega\in\mathfrak{F}_+$) befriedigt mit der Funktion $\omega(t, x, y, z)\equiv x-L(t)\,y$, dabei ist $\omega\in M$. Die Ungleichungen (25) gehen damit über in die Differentialungleichung $\varrho'-L(t)\varrho>\delta(t)$ mit $\varrho(0)>\varepsilon$. Für $\gamma>0$ und $\delta\in C(I)$ ist

$$\varrho(t)=\gamma(1+t)+\sigma(t)$$

mit

$$\sigma(t)=\exp\int_0^t L(s)\,ds\left[\varepsilon+\int_0^t\delta(s)\exp-\int_0^s L(\tau)\,d\tau\,ds\right] \tag{29}$$

offenbar eine Lösung, für $\gamma\to 0$ folgt daraus die bekannte Fehlerabschätzung

$$|v(t)-u(t)|\leqq\sigma(t) \quad \text{mit} \quad \sigma(t) \text{ aus (29)} \tag{30}$$

für alle Lösungen $u(t)$ von (27) und alle Näherungsfunktionen $v(t)$, für die $|v(0)-u(0)|\leqq\varepsilon$ und $|v'-g(t,v)|\leqq\delta(t)$ in I_0 gilt.

Es soll dabei besonders darauf hingewiesen werden, daß es zur *beidseitigen* Abschätzung der Differenz $v-u$ nach (29), (30) genügt, eine *einseitige* Lipschitz-Abschätzung (28) vorauszusetzen. Damit sind auch negative Funktionen $L(t)$ zugelassen (vgl. etwa M. MÜLLER [2]), in diesem Falle ist das Fehlerverhalten besonders „gutartig“. In der Literatur wird statt (28) oft

$$|f(t,y)-f(t,\bar{y})|\leqq\bar{L}(t)\,|y-\bar{y}| \quad \text{für} \quad t\in I_0, \qquad -\infty<y,\bar{y}<+\infty$$

gefordert, wodurch für $L \leqq 0$ die Abschätzung (30) unter Umständen ganz erheblich verschlechtert wird.

b) Man betrachtet die (nichtlineare) Volterrasche Integralgleichung zweiter Art

$$u(t) - \int_0^t K\big(s, t, u(s)\big)\, ds = g(t), \tag{31}$$

die in (2) vorgeschriebene Anfangsbedingung $u(0) = \eta$ ist mit $g(0) = \eta$ in (31) implizit schon enthalten. Man verwendet also etwa die Funktion $f(t, x, y, z) \equiv y - z - g(t)$ und den Operator

$$F u \equiv \int_0^t K\big(s, t, u(s)\big)\, ds.$$

Es ist $F \in \mathfrak{F}$, wenn K integrierbar ist für $s, t \in I$ und $u \in C(I)$. Genügt der Kern $K(s, t, z)$ für $s, t \in I$ der Lipschitz-Bedingung

$$K(s, t, z) - K(s, t, \bar{z}) \leqq L(s)\, N(t)\, (z - \bar{z}) \quad \text{für alle} \quad z \geqq \bar{z}, \tag{32}$$

mit einer in I stetigen Funktion $N(t)$, dann ist die Ungleichung (24) befriedigt mit der Funktion $\omega(t, x, y, z) \equiv y - N(t)\, z$ und dem Operator

$$\Omega u \equiv \int_0^t L(s)\, u(s)\, ds.$$

Ist $L \geqq 0$ und $N \geqq 0$ auf I, dann gilt $\omega \in M$ und $\Omega \in \mathfrak{F}_+$, also kann Satz 4 angewandt werden. Die Ungleichung (25) geht über in die Volterrasche Integralungleichung zweiter Art

$$\varrho(t) - N(t) \int_0^t L(s)\, \varrho(s)\, ds > \delta(t) \quad \text{mit} \quad \varrho(0) > \varepsilon, \tag{33}$$

wo ε und $\delta(t)$ durch

$$\begin{aligned} &|v(0) - u(0)| \leqq \varepsilon, \\ &\left| v(t) - \int_0^t K\big(s, t, v(s)\big)\, ds - g(t) \right| \leqq \delta(t) \quad \text{in } I_0 \end{aligned} \tag{34}$$

definiert sind.

Für $\gamma > 0$, $\delta \in C(I)$ und $N(0) \neq 0$ ist

$$\varrho(t) = \gamma N(t) \exp \int_0^t L(s)\, N(s)\, ds + \sigma(t)$$

mit

$$\begin{aligned} \sigma(t) = \delta(t) + N(t) \exp \int_0^t L(s)\, N(s)\, ds \Big\{ &\big[\mathrm{Max}\big(\varepsilon, \delta(0)\big) - \delta(0)\big]/N(0) + \\ &+ \int_0^t \delta(s)\, L(s) \exp - \int_0^s L(r)\, N(r)\, dr\, ds \Big\} \end{aligned} \tag{35}$$

eine Lösung von (33). Mit $\gamma \to 0$ folgt daraus die Fehlerabschätzung

$$|v(t) - u(t)| \leqq \sigma(t) \quad \text{mit} \quad \sigma(t) \text{ aus (35)} \tag{36}$$

für alle Lösungen $u(t)$ von (31) und alle Näherungslösungen $v(t)$, die den Abschätzungen (34) genügen.

Führt man die Differentialgleichung (27) durch Integration über in die Volterrasche Integralgleichung zweiter Art

$$u(t) - \int_0^t g\big(s, u(s)\big)\, ds = \eta$$

entsprechend zu (31), so ist hier nur der Fall $L \geqq 0$ zulässig, d.h. der oben erwähnte Vorteil bei den Fehlerabschätzungen geht bei dieser Transformation verloren!

c) Man betrachtet das Anfangswertproblem mit der Fredholmschen Integro-Differentialgleichung

$$u'(t) - g\big(t, u(t)\big) - \int_0^T K\big(s, t, u(s)\big)\, ds = 0, \qquad u(0) = \eta, \tag{37}$$

wählt also

$$f(t, x, y, z) \equiv x - g(t, y) - z \quad \text{und} \quad F u \equiv \int_0^T K\big(s, t, u(s)\big)\, ds.$$

Ist K integrierbar für $s, t \in I$, $u \in C(I)$, so gilt $F \in \mathfrak{F}$. Es sei

$$g(t, y) - g(t, \bar{y}) \leqq 2L(y - \bar{y}) \quad \text{für} \quad t \in I_0 \quad \text{und alle} \quad y \geqq \bar{y}$$

und für $t \in I$ gelte

$$\int_0^T \big[K\big(s, t, w(s)\big) - K\big(s, t, \overline{w}(s)\big)\big]\, ds \leqq N \int_0^t [w(s) - \overline{w}(s)]\, ds \tag{38}$$

für alle Funktionen $w, \overline{w} \in C(I)$ mit $w(t) \geqq \overline{w}(t)$ auf I. Dann wird (24) befriedigt durch die Funktion $\omega(t, x, y, z) \equiv x - 2Ly - Nz$, wenn $\Omega u \equiv \int_0^t u(s)\, ds$ gesetzt wird. Es ist $\Omega \in \mathfrak{F}_+$ und $\omega \in M$, wenn $N \geqq 0$ gilt. Ist $\delta(t) \leqq \delta$, dann geht (25) über in die Ungleichungen

$$\varrho' - 2L\varrho - N \int_0^t \varrho(s)\, ds > \delta, \qquad \varrho(0) > \varepsilon,$$

die für alle $\gamma > 0$ durch

$$\varrho(t) = \gamma \exp(\lambda + 1)\, t + \sigma(t)$$

mit

$$\sigma(t) = \begin{cases} \varepsilon + \delta t & \text{für} \quad L^2 + N = 0 \\ [(\varepsilon\lambda + \delta) \exp \lambda t - (\varepsilon\mu + \delta) \exp \mu t]/(\lambda - \mu) & \text{für} \quad L^2 + N \neq 0 \end{cases} \tag{39}$$

und

$$\lambda = L + \sqrt{L^2 + N}, \qquad \mu = L - \sqrt{L^2 + N}$$

befriedigt werden. Für $\gamma \to 0$ findet man dann die Fehlerabschätzung

$$|v(t) - u(t)| \leqq \sigma(t) \quad \text{auf } I \text{ mit } \sigma \text{ aus (39)} \tag{40}$$

für alle Lösungen $u(t)$ von (37) und alle Näherungslösungen $v(t)$, deren Anfangsfehler durch ε und deren Defekt durch δ beschränkt wird. Wegen $\lambda \geqq 0$ nimmt die Schrankenfunktion $\sigma(t)$ nur für $L = N = \delta = 0$ oder für $\varepsilon = \delta = 0$ nicht zu.

Für beliebige Funktionen f und Operatoren $F \in \mathfrak{F}$ sind die Voraussetzungen des Satzes 4 im allgemeinen nicht erfüllbar, insbesondere braucht es keine

Funktion ω und keinen Operator Ω zu geben, so daß (24) befriedigt ist. Das zeigt das folgende

Gegenbeispiel 3. Es sei $f(t, x, y, z) \equiv y^2$, dann geht die Ungleichung (24) über in

$$w^2 - \overline{w}^2 = (w - \overline{w})(2\overline{w} + w - \overline{w}) \geqq \omega(t, \ldots, w - \overline{w}, \ldots)$$

für alle Funktionen $w, \overline{w} \in C(I)$ mit $w \geqq \overline{w}$. Wäre $\overline{w}$ eine festgehaltene Funktion, so könnte etwa $\omega(t, x, y, z) = 2wy + y^2$ gesetzt werden; für beliebige Funktionen $w \in C(I)$ ist diese Ungleichung jedoch offenbar nicht zu befriedigen.

Im folgenden Abschätzungssatz 5 sind daher die in Satz 4 noch benötigten Voraussetzungen sehr stark reduziert. Dabei werden besonders die speziellen Eigenschaften der Näherungslösung $v(t)$ noch weiter ausgenützt; außerdem zeigt es sich, daß man durch einseitige Abschätzungen oftmals zu günstigeren Fehlerschranken gelangt. Es gilt der

Satz 5. *Die Funktion $f(t, x, y, z)$ sei in $t \in I_0$, $-\infty < x, y, z < +\infty$ erklärt, der Operator F sei $\in \mathfrak{F}$. Die Funktionen $u, v \in Z(I)$ seien eine Lösung und eine Näherungslösung zu dem Anfangswertproblem* (1), (2) *und zwar mögen mit geeigneten Konstanten $\varepsilon, \overline{\varepsilon}$ und in I_0 erklärten Funktionen $\delta(t)$, $\overline{\delta}(t)$ die Abschätzungen*

$$-\overline{\varepsilon} \leqq v(0) - u(0) \leqq \varepsilon, \tag{41}$$

$$f(t, u', u, Fu) = 0, \qquad -\overline{\delta}(t) \leqq f(t, v', v, Fv) \leqq \delta(t) \quad \text{in } I_0 \tag{42}$$

gelten.

Die Funktionen $\omega, \overline{\omega} \in M[\overline{M}]$, die Operatoren $\Omega, \overline{\Omega} \in \mathfrak{F}_+[\overline{\mathfrak{F}}_+]$ und die Funktionen $\varrho, \overline{\varrho}\, Z(I)$ mögen den folgenden beiden Systemen aus je drei Ungleichungen genügen, es gelte nämlich

$$f(\bar{t}, v', v, Fv) - f\big(\bar{t}, v' - \varrho', v - \varrho, F(v - \varrho)\big) \geqq \omega(\bar{t}, \varrho', \varrho, \Omega\varrho) \tag{43}$$

für alle $\bar{t} \in I_0$ mit $v = u + \varrho$, falls für $0 \leqq t < \bar{t}$ stets $v - u < \varrho$ bleibt,

$$\omega(t, \varrho', \varrho, \Omega\varrho) > [\geqq]\, \delta(t) \quad \text{in} \quad I_0, \qquad \varrho(0) > \varepsilon; \tag{44}$$

$$f\big(\bar{t}, v' + \overline{\varrho}', v + \overline{\varrho}, F(v + \overline{\varrho})\big) - f(\bar{t}, v', v, Fv) \geqq \overline{\omega}(\bar{t}, \overline{\varrho}', \overline{\varrho}, \overline{\Omega}\overline{\varrho}) \tag{45}$$

für alle $\bar{t} \in I_0$ mit $v = u - \overline{\varrho}$, falls für $0 \leqq t < \bar{t}$ stets $u - v < \overline{\varrho}$ bleibt,

$$\overline{\omega}(t, \overline{\varrho}', \overline{\varrho}, \overline{\Omega}\overline{\varrho}) > [\geqq]\, \overline{\delta}(t) \quad \text{in } I_0, \qquad \overline{\varrho}(0) > \overline{\varepsilon}. \tag{46}$$

Dann gilt die Abschätzung

$$-\overline{\varrho}(t)\,[\leqq] < v(t) - u(t) < [\leqq]\, \varrho(t) \tag{47}$$

auf ganz I.

Der *Beweis* verläuft mit Satz 2 [Satz 3] ganz entsprechend zum Beweise des Satzes 4. Satz 4 ist ein Sonderfall des allgemeineren Satzes 5.

Offenbar ist es durch die Verwendung der Näherungslösung v und sogar der (im allgemeinen unbekannten) Lösung u in den Ungleichungen (43) und (45) bei konkreten Beispielen häufig möglich, zu erheblich günstigeren Funktionen ω und $\overline{\omega}$ als in Satz 4 und damit auch zu günstigeren Schrankenfunktionen ϱ und $\overline{\varrho}$ zu gelangen. Dies soll deutlich gemacht werden durch das folgende

Beispiel 6. Es soll die Lösung★ $u(t) \in Z(I)$ der Volterraschen Integro-Differentialgleichung

$$(1+t)\,u' + 3u - \sin u - \frac{1}{t(1+t)} \int_0^t \cos u(s)\,ds = \frac{3}{1+t} \tag{48}$$

unter der Anfangsbedingung $u(0)=0$ im Intervall $0 \leqq t < \infty$ abgeschätzt werden.

Man wählt T beliebig groß, $Fu \equiv \int_0^t \cos u(s)\,ds$ und

$$f(t,x,y,z) \equiv (1+t)\,x + 3y - \sin y - z/t(1+t) - 3/(1+t).$$

Offenbar ist $f \notin M$ und $F \notin \mathfrak{F}_+$, so daß eine Abschätzung durch Ober- und Unterfunktionen mit den Methoden von Teil I nicht unmittelbar möglich ist. Zur Bestimmung einer Näherungsfunktion $v(t)$ ersetzt man $\sin u$ durch u und $\cos u$ durch 1 in (48) und findet damit die explizite Näherungslösung

$$v(t) = 4t/(1+t)^2.$$

Es ist

$$v(0) = u(0) = 0,$$

$$f(t, v', v, Fv) \equiv d(t)$$

mit

$$d(t) = v - \sin v + t^{-1}(1+t)^{-1}\int_0^t (1-\cos v)\,ds$$

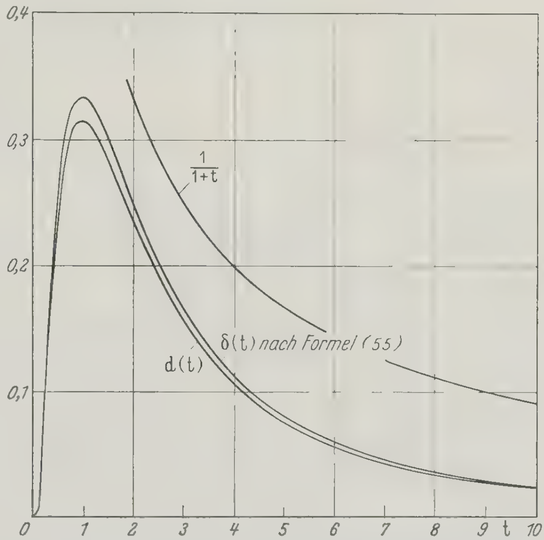

Abb. 1. Zu Beispiel 6: Defekt $d(t)$ und Defektschranke $\delta(t)$

(vgl. Abb. 1); man kann daher in (41) und (42) etwa $\varepsilon = \bar{\varepsilon} = 0$, $-\bar{\delta}(t) = \delta(t) = d(t)$ wählen.

Die Ungleichung (43) wird zu

$$f(t,v',\ldots) - f(t,v'-\varrho',\ldots) = (1+t)\,\varrho' + 3\varrho - 2\sin\varrho/2\cos(v-\varrho/2) +$$
$$+ 2t^{-1}(1+t)^{-1}\int_0^t \sin\varrho/2\sin(v-\varrho/2)\,ds \geqq (1+t)\,\varrho' + 2\varrho,$$

falls nur

$$\varrho \geqq 0 \tag{49}$$

und

$$2v - \varrho \geqq 0 \tag{50}$$

bleibt in $0 \leqq t < \infty$. Diese Eigenschaft läßt sich — für die noch zu bestimmende Funktion $\varrho(t)$ — später nachweisen, also kann $\omega(t,x,y,z) \equiv (1+t)\,x + 2y$ mit $\omega \in M$ gewählt werden. Damit geht (44) über in die Differentialungleichung $(1+t)\,\varrho' + 2\varrho > d(t)$ mit $\varrho(0) > 0$; eine Lösung ist $\varrho(t) = \gamma(1+t) + \sigma(t)$ mit

$$\sigma(t) = (1+t)^{-2}\int_0^t (1+s)\,d(s)\,ds \tag{51}$$

★ Ein Existenzbeweis kann — etwa mit dem Iterationsverfahren — leicht geführt werden, die Eindeutigkeit einer Lösung wird in Beispiel 7 nachgewiesen.

für alle Zahlen $\gamma > 0$ (vgl. Abb. 2). Wegen $d(t) \geqq 0$ (s. Abb. 1) ist $\sigma \geqq 0$ und damit auch $\varrho \geqq 0$, also ist (49) erfüllt. Aus $d(t) \leqq (1+t)^{-1}$ (s. Abb. 1) folgt $\sigma(t) \leqq 4v(t)$, also auch $\varrho(t) < 2v(t)$ auf I, wenn nur γ hinreichend klein gewählt wird; mithin ist auch (50) befriedigt. Für $\gamma \to 0$ findet man aus (47) damit schließlich $v(t) - u(t) \leqq \sigma(t)$.

Zur Abschätzung von $v - u$ nach unten betrachtet man die aus (45) folgende Ungleichung

$$f(t, v' + \bar{\varrho}', \ldots) - f(t, v', \ldots) = (1+t)\bar{\varrho}' + 3\bar{\varrho} - 2\sin\bar{\varrho}/2 \cos(v + \bar{\varrho}/2) +$$
$$+ 2t^{-1}(1+t)^{-1}\int_0^t \sin\bar{\varrho}/2 \sin(v - \bar{\varrho}/2)\, ds \geqq (1+t)\bar{\varrho}' + 3\bar{\varrho} - a(t)$$

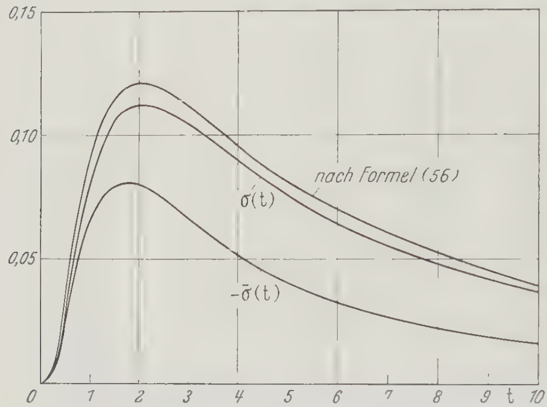

Abb. 2. Zu Beispiel 6: Fehlerschranken $\sigma(t)$ und $-\bar{\sigma}(t)$

mit

$$a(t) = 2t^{-1}(1+t)^{-1} \times \int_0^t \sin\varrho/2 \sin(v - \varrho/2)\, ds,$$

wenn $-\bar{\varrho}(t) < \varrho(t)$ ist und falls nur

$$\bar{\varrho} \leqq 0 \tag{52}$$

und

$$2v + \bar{\varrho} \leqq \pi \tag{53}$$

bleibt in $0 \leqq t < \infty$. Auch diese Eigenschaften werden sich anschließend nachweisen lassen. Also kann

$$\overline{\omega}(t, x, y, z) \equiv (1+t)x + 3y - a(t)$$

gewählt werden, die Ungleichung (46) geht damit über in die Differentialungleichung

$$(1+t)\bar{\varrho}' + 3\bar{\varrho} - a(t) > -d(t) \quad \text{mit} \quad \bar{\varrho}(0) > 0;$$

eine Lösung ist $\bar{\varrho}(t) = (1+t)\gamma + \bar{\bar{\sigma}}(t)$ für alle $\gamma > 0$, wenn

$$-\bar{\bar{\sigma}}(t) = (1+t)^{-3}\int_0^t (1+s)^2 [d(s) - a(s)]\, ds$$

gilt. Wie man leicht nachprüft, sind die Ungleichungen (52) und (53) für alle $t \geqq 0$ erfüllt, also folgt aus (47) für $\gamma \to 0$ die Abschätzung $-\bar{\sigma}(t) \leqq v(t) - u(t)$ mit

$$-\bar{\sigma}(t) = (1+t)^{-3}\int_0^t \left[(1+s)^2 d(s) - 2(1+s)s^{-1}\int_0^s \sin\sigma/2 \sin(v - \sigma/2)\, d\tau\right] ds \tag{54}$$

(vgl. Abb. 2).

Die Schranken $v(t) - \sigma(t)$ und $v(t) + \bar{\sigma}(t)$ begrenzen in Abb. 3 einen (schraffierten) Streifen, der die exakte Lösung $u(t)$ von (48) enthält. Man entnimmt Abb. 3 das verblüffende Ergebnis, daß dieser Streifen *außerhalb* der verwendeten Näherungslösung $v(t)$ verläuft, so daß also die Fehlerabschätzung nach Satz 5 hier sogar zu einer Korrektur dieser Näherung benutzt werden kann, indem man etwa $\bar{v} = v + (\bar{\sigma} - \sigma)/2$ als neue und bessere Näherung für u verwendet. Die Berechnung des Defektes $\bar{d}(t)$ zu $\bar{v}(t)$ und neuerliche Fehlerabschätzung für $\bar{v}$ aus (51) und (54) würde einen neuen, schmaleren Streifen ergeben, innerhalb

dessen die exakte Lösung $u(t)$ gelegen ist; dieses Vorgehen ließe sich noch iterieren. Ebenfalls ließe sich eine Verbesserung der Fehlerabschätzung erzielen, wenn man die noch ziemlich groben Abschätzungen, die oben zu den Funktionen ω und $\overline{\omega}$ führten, durch Verwendung der jetzt schon recht genau bekannten Schranken σ und $\overline{\sigma}$ verfeinern würde. Da in dem vorliegenden Beispiel nur das Prinzip einer solchen Fehlerabschätzung gezeigt werden sollte, wurde auf die numerische Durchführung dieser Vorschläge verzichtet.

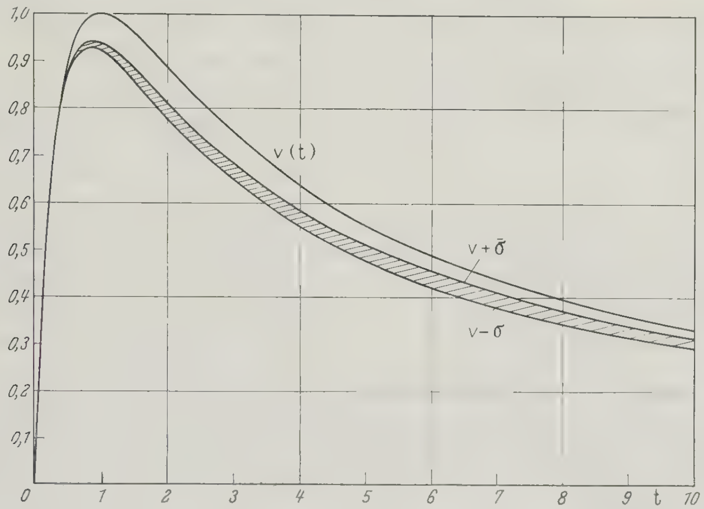

Abb. 3. Zu Beispiel 6: Näherungslösung $v(t)$ und Schranken $v+\overline{\sigma}$ bzw. $v-\sigma$ für die exakte Lösung $u(t)$

Schließlich ist noch zu sehen, daß sich leicht auch geschlossene Ausdrücke aus elementaren Funktionen für σ und $\overline{\sigma}$ angeben lassen, z.B. folgt aus

$$v - \sin v \leqq v^3/6, \qquad 1 - \cos v \leqq v^2/2$$

die Schranke

$$d(t) \leqq \tfrac{8}{3} t^2 (1 + 6t + t^2)(1+t)^{-6} \equiv \delta(t) \tag{55}$$

(vgl. Abb. 1) und damit nach (51) die Formel

$$\sigma(t) \leqq \tfrac{4}{3}[2(1+t)^{-2} \ln(1+t) - (1+t)^{-2} - 4(1+t)^{-3} + \\ + 11(1+t)^{-4} - 8(1+t)^{-5} + (1+t)^{-6}] \tag{56}$$

(vgl. Abb. 2).

III. Eindeutigkeitssätze

Aus den Fehlerabschätzungen des Abschnitts II folgen in vielen Fällen Einzigkeitssätze. So haben etwa die in den Beispielen 5a) bis 5c) betrachteten Anfangswertaufgaben jeweils höchstens eine einzige Lösung, denn unter den Lipschitz-Bedingungen (28), (32) und (38) strebt in den Abschätzungen (30), (36) und (40) die rechte Seite gegen Null, also strebt $v(t) \to u(t)$, wenn $\varepsilon \to 0$ und $\delta \to 0$ geht. Etwas allgemeiner ist der folgende Eindeutigkeitssatz:

Satz 6. *Die Funktion $f(t, x, y, z)$ sei in $t \in I_0$, $-\infty < x, y, z < +\infty$ erklärt, der Operator F sei $\in \mathfrak{F}$. Zu $\gamma > 0$ und je zwei Lösungen $u, v \in Z$ des Anfangswertproblems* (1), (2) *gebe es stets eine Funktion $\omega \in M[\overline{M}]$ und einen Operator $\Omega \in \mathfrak{F}_+[\overline{\mathfrak{F}}_+]$*

derart, daß
$$\omega\left(\bar{t}, v' - u', v - u, \Omega(v-u)\right) \leqq 0 \tag{57}$$
ist für alle Stellen $\bar{t} \in I_0$ *mit* $v(\bar{t}) - u(\bar{t}) = \gamma$ *und* $v(t) - u(t) < \gamma$ *in* $0 < t < \bar{t}$. *Weiter existiere eine Funktion* $\varrho \in Z$, *für die* $0 \leqq \varrho \leqq \gamma$ *in* I_0 *und*
$$\omega(t, \varrho', \varrho, \Omega\varrho) > [\geqq]\ 0 \quad in\ I_0 \tag{58}$$
und
$$\varrho(0+) > v(0+) - u(0+) \tag{59}$$
ist. Dann besitzt das Anfangswertproblem (1), (2) *höchstens eine einzige Lösung* $u \in Z$.

Der *Beweis* verläuft ganz analog zu demjenigen von Satz 4.

Bemerkung. Die Voraussetzung (57) ist z.B. erfüllt, wenn $f(t, x, y, z)$ einer Abschätzung (24) genügt, denn dann ist für zwei Lösungen u, v von (1), (2) stets
$$0 = f(t, v', v, Fv) - f(t, u', u, Fu) \geqq \omega\left(t, v' - u', v - u, \Omega(v-u)\right).$$

Beispiel 7. Als einfache Anwendung von Satz 6 soll zunächst die Einzigkeit jeder Lösung der in Beispiel 6 betrachteten Integro-Differentialgleichung (48) nachgewiesen werden. Offenbar gilt für die dort verwendete Funktion f und den Operator F die Abschätzung
$$f(t, x, w, Fw) - f(t, \bar{x}, \bar{w}, F\bar{w}) \geqq (1+t)(x-\bar{x}) + 2(w-\bar{w}) - \frac{1}{t(1+t)} \int_0^t [w(s) - \bar{w}(s)]\, ds$$
für beliebige Zahlen x, $\bar{x}$ und alle Funktionen $w, \bar{w} \in C(I)$ mit $w \geqq \bar{w}$. Also ist die Bedingung (57) erfüllt mit der Funktion $\omega(t, x, y, z) \equiv (1+t)\,x + 2y - z/t(1+t)$ und mit dem Operator
$$\Omega w \equiv \int_0^t w(s)\, ds;$$
dabei ist $\omega \in \overline{M}$ und $\Omega \in \overline{\mathfrak{F}}_+$. Setzt man nun $\varrho(t) = \gamma$ in I, dann ist $\omega(t, \varrho', \varrho, \Omega\varrho) = \gamma(1+2t)/(1+t) > 0$ in I, also befriedigt $\varrho(t)$ die Ungleichung (58). Weiter gilt $\varrho(0) = \gamma > 0 = v(0) - u(0)$ für je zwei Lösungen u, v von (48) mit einer beliebigen Anfangsbedingung $u(0) = \eta$, also ist auch (59) erfüllt. Mithin besitzt (48) unter beliebigen Anfangsbedingungen nach Satz 6 höchstens eine einzige Lösung $u \in Z$.

Beispiel 8 (verallgemeinerte Nagumo-Bedingungen).

a) Gewöhnliche Differentialgleichung erster Ordnung. Man betrachtet
$$u' - g(t, u) = 0, \qquad u(0) = \eta, \tag{27}$$
also $f(t, x, y, z) \equiv x - g(t, y)$, F beliebig. Die reelle Zahl η sei beliebig gewählt, die Funktion $g(t, y)$ sei definiert für $t \in I$, $-\infty < y < +\infty$ und sei noch stetig an der Stelle $t = 0$, $y = \eta$. Dann gilt für jede Lösung $u \in Z$ des Anfangswertproblems (27) stets $u(t) = \eta + t g(t, \eta) + o(t)$ für $t \to 0$, also ist dann
$$v(t) - u(t) = o(t) \quad \text{für} \quad t \to 0 \tag{60}$$
für je zwei Lösungen u, v von (27). Weiter genüge die Funktion g der verallgemeinerten Nagumo-Bedingung*.
$$g(t, y) - g(t, \bar{y}) \leqq \left[\frac{1}{t} + L(t)\right](y - \bar{y}) \tag{61}$$

* Diese Bedingung verdanke ich Herrn Doz. Dr. W. WALTER.

für $t \in I_0$ und alle $y \geqq \bar{y}$. Dabei sei die Funktion $L(t)$ in I_0 stetig und das Integral

$$\int_0^t L(s)\,ds$$

existiere (eventuell uneigentlich) für $t \in I$. Dann genügt die Funktion $\omega(t, x, y, z) \equiv x - \left(\frac{1}{t} + L(t)\right) y$ der Ungleichung (57) für einen beliebigen Operator $\Omega \in \widetilde{\mathfrak{F}}_+$ und es ist $\omega \in M$. Ist $\gamma > 0$ fest gewählt, so liegt die Funktion

$$\varrho(t) \equiv \beta\, t(1+t) \exp \int_0^t L(s)\,ds$$

für hinreichend kleine Zahlen $\beta > 0$ stets in $0 \leqq \varrho \leqq \gamma$. Weiter ist $\varrho \in Z$ und genügt wegen

$$\omega(t, \varrho', \varrho, \ldots) = \beta\, t \exp \int_0^t L(s)\,ds > 0 \quad \text{in } I_0$$

der Bedingung (58). Schließlich ist noch $\varrho(t) = \beta t + o(t)$ für $t \to 0$, also ist $\varrho(0+) > v(0+) - u(0+)$ nach (60), womit auch Ungleichung (59) befriedigt ist. Damit hat das Anfangswertproblem (27) unter der Bedingung (61) höchstens* eine einzige Lösung $u \in Z$.

In dieser Aussage ist der oben erwähnte, aus Beispiel 5a) folgende Eindeutigkeitssatz *nicht* enthalten, obwohl natürlich die Bedingung (61) auch die Lipschitz-Bedingung (28) mit umfaßt. In Beispiel 5a) war nämlich über die Funktion $g(t, y)$ nur vorausgesetzt, daß g für $t \in I_0$, $-\infty < y < \infty$ definiert sein sollte, während beim vorliegenden Satz g noch bei $t=0$ definiert und an der Stelle $t=0$, $y=\eta$ sogar noch stetig sein soll.

b) Volterrasche Integralgleichung zweiter Art. Man betrachtet die nichtlineare Integralgleichung

$$u(t) - \int_0^t K\big(s, t, u(s)\big)\,ds = g(t). \tag{31}$$

Es sei

$$f(t, x, y, z) \equiv y - z - g(t), \qquad F u \equiv \int_0^t K\big(s, t, u(s)\big)\,ds$$

gewählt. Die Kernfunktion $K(s, t, u)$ sei integrierbar für $s, t \in I$, $u \in C(I)$; die Funktion $g(t)$ sei stetig auf I und die vordere Ableitung $d_+ g(t)/dt$ existiere bei $t=0$. Weiter genüge K der Bedingung

$$K(s, t, y) - K(s, t, \bar{y}) \leqq M(s)\, N(t)\, (y - \bar{y}) \tag{62}$$

für $s, t \in I_0$ und $y \geqq \bar{y}$, wobei die Funktion $M(s) \geqq 0$ und auf I_0 definiert, $N(t) > 0$ und auf I stetig sein soll; und in I_0 gelte noch

$$M(t)\, N(t) \leqq \frac{1}{t} + L(t) \tag{63}$$

mit einer in I_0 stetigen, auf I (eventuell uneigentlich) integrierbaren Funktion $L(t)$. $L \equiv 0$ entspricht offenbar der üblichen Nagumo-Bedingung.

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* Unter den genannten geringen Voraussetzungen an die Funktion g ist die Existenz einer Lösung von (27) natürlich nicht allgemein gesichert.

Zur Anwendung des Satzes 6 setzt man dann

$$\omega(t, x, y, z) \equiv y - z \quad \text{mit} \quad \omega \in M \quad \text{und} \quad \Omega u \equiv N(t) \int_0^t \left[\frac{1}{s} + L(s)\right] u(s)\, ds.$$

Der Operator Ω ist zwar monoton, wegen des Terms $1/s$ im Integranden jedoch nicht auf alle Funktionen $u \in C(I)$ anwendbar. Immerhin existiert jedoch Ωw sicher für alle Funktionen $w \in C(I)$ mit $w(t) = O(t)$ für $t \to 0$. Offenbar genügt es nun schon zum Beweis von Satz 6, wenn Ω allein definiert ist für die Differenz $v - u$ zweier Lösungen u, v von (31) und für die gesuchte Funktion $\varrho(t)$. Wegen

$$u(t) = g(0) + t[g'_+(0) + K(0, 0, g(0))] + o(t)$$

ist jedoch

$$v(t) - u(t) = o(t) \quad \text{für} \quad t \to 0 \tag{64}$$

für die Differenz zweier beliebiger Lösungen $u, v \in Z$ von (31), also ist $\Omega(v-u)$ auf I definiert und stetig. Wegen (62) und (63) ist dann die Ungleichung (57) erfüllt für alle Lösungen $u, v \in Z$ von (31).

Man setzt nun

$$\varrho(t) = \beta\, t\, N(t) \exp \int_0^t L(s)\, ds.$$

Ist $\beta > 0$ hinreichend klein und $\gamma > 0$ gewählt, so gilt $0 \leqq \varrho(t) \leqq \gamma$ für alle $t \in I$. Wegen $\varrho(t) = \beta t + o(t)$ ist $\varrho(0+) > v(0+) - u(0+)$ nach (64), also befriedigt ϱ die Bedingung (59). Weiter ist damit der Operator Ω anwendbar auf ϱ und es gilt

$$\omega(t, \ldots, \varrho, \Omega\varrho) = \beta N(t) \int_0^t \exp \int_0^s L(\tau)\, d\tau\, ds > 0 \quad \text{in } I_0,$$

d.h. auch (58) ist erfüllt. Damit hat die Integralgleichung (31) unter der Bedingung (62), (63) höchstens eine einzige Lösung $u \in C(I)$.

Die Einzigkeitsbedingung (62), (63) findet sich für $N \equiv 1$ und $L \equiv 0$ bei T. SATŌ [6], S. 282, allgemein ist sie in dessen Théorème 13 enthalten. Auch hier ist der aus Beispiel 5b) fließende Eindeutigkeitssatz nicht in dem soeben abgeleiteten Ergebnis enthalten, da in Beispiel 5b) die Voraussetzungen über K und f milder sind.

c) Integro-Differentialgleichungen. Gegeben sei die Integro-Differentialgleichung

$$u'(t) - g(t, u(t)) - \int_0^T K(s, t, u(s))\, ds = 0 \tag{37}$$

mit der Anfangsbedingung $u(0) = \eta$. Es sei

$$f(t, x, y, z) \equiv x - g(t, y) - z \quad \text{und} \quad Fu \equiv \int_0^T K(s, t, u(s))\, ds$$

gewählt; es sei $F \in \mathfrak{F}$, das ist z.B. der Fall, wenn die Kernfunktion $K(s, t, u)$ integrierbar für $s, t \in I$, $u \in C(I)$ ist. Die Funktion $g(t, y)$ sei definiert für $t \in I$, $-\infty < y < \infty$, stetig an der Stelle $t = 0$, $y = \eta$ und genüge der Lipschitz-Bedingung

$$g(t, y) - g(t, \bar{y}) \leqq L(t)\,(y - \bar{y}) \quad \text{für} \quad t \in I_0, \qquad y \geqq \bar{y} \tag{28}$$

mit einer in I_0 stetigen Funktion $L(t)$. Weiter gelte

$$\int_0^T \left[K\big(s,t,w(s)\big)-K\big(s,t,\overline{w}(s)\big)\right]ds \leqq N(t)\int_0^t M(s)\left[w(s)-\overline{w}(s)\right]ds \qquad (65)$$

für alle $t\in I$ und alle Funktionen $w, \overline{w}\in C(I)$, für die $w\geqq\overline{w}$ auf I gilt. Dabei sei $N, M\in C(I_0)$ mit $N\geqq 0$, $M\geqq 0$ auf I und für alle $t\in I$ existiere das Integral

$$\int_0^t s\,M(s)\,ds \quad \text{(eventuell uneigentlich).}$$

Schließlich sei noch

$$N(t)+t\,L(t)\leqq 1+t^2 M(t) \quad \text{in } I_0. \qquad (66)$$

Die Bedingung (66) ist z.B. erfüllt, wenn L und N in I_0 je einer Abschätzung $L(t)\leqq\lambda/t+l$, $N(t)\leqq\nu+t\,n$ genügen, wobei die Konstanten λ, l, ν, n die Ungleichungen $\lambda+\nu\leqq 1$, $l+n\leqq t\,M(t)$ in I befriedigen. Ist $\nu=l=0$, so ist dies die Nagumo-Bedingung für $L(t)$.

Man setzt nun $\omega(t,x,y,z)\equiv x-L(t)\,y-N(t)\,z$ in I_0 und

$$\Omega w\equiv\int_0^t M(s)\,w(s)\,ds.$$

Wegen (28) und (65) ist dann (57) erfüllt. Es ist $\omega\in M$, der Operator Ω ist monoton, jedoch ist Ω wie im Falle b) unter Umständen nicht auf alle Funktionen $w\in C(I)$ anwendbar. Genau wie dort überlegt man sich, daß es ausreicht, wenn Ω allein definiert ist für die Differenz $v-u$ zweier Lösungen $u, v\in Z$ von (37) und für die gesuchte Funktion ϱ.

Es ist

$$u(t)=\eta+t\left[g(0,\eta)+\int_0^T K(s,0,\eta)\,ds\right]+o(t),$$

also gilt

$$v(t)-u(t)=o(t) \quad \text{für} \quad t\to 0, \qquad (67)$$

mithin ist $\Omega(v-u)$ auf I definiert und stetig.

Man setzt nun

$$\varrho(t)=\beta\,t(1+t)\exp\int_0^t s(1+s)\,M(s)\,ds$$

mit einer hinreichend kleinen Zahl $\beta>0$. Es ist $\varrho\in Z$ und $\varrho(t)=\beta t+o(t)$, also gilt $\varrho(0+)>v(0+)-u(0+)$ nach (67), folglich ist die Bedingung (59) erfüllt. Mit (66) und wegen $M\geqq 0$, $N\geqq 0$ ist weiter

$$\begin{aligned}\omega(t,\varrho',\varrho,\Omega\varrho)&=\frac{\varrho}{t}\left(1+t^2M-t\,L-N\right)+\frac{\varrho}{1+t}\left(1+t^2(1+t)\,M+N\right)+\beta N\\&\geqq\varrho/(1+t)>0 \quad \text{in } I_0,\end{aligned}$$

also befriedigt ϱ auch noch (58). Damit besitzt das Anfangswertproblem (37), (2) unter den Bedingungen (28), (65), (66) höchstens eine einzige Lösung $u\in Z$.

Analog zu oben ist in diesen Einzigkeitskriterien das Ergebnis von Beispiel 5c) *nicht* als Sonderfall enthalten. Dagegen erhält man für $N\equiv 0$ den Eindeutigkeitssatz von Beispiel 8a).

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